Are fractals really just a work of art?

Fractals – a sophisticated blend of simplicity and complexity to produce dumbfounding patterns, appreciated by all mathematicians. A fractal is defined as a self-similar geometrical figure, where the pattern is duplicated every time it is zoomed in. These marvels can be created using computer graphics, or could even be painted. A brain could concentrate at a single fractal for a while, and this exposure can be said to reduce stress levels by up to 60 percent. Nevertheless, are fractals just appreciative patterns that we humans like to glare upon, or do they serve a mathematical purpose?

The father of fractal geometry, Benoit Mandelbrot, was said to have a remarkable geometric intuition, which paved him into giving unique insight to mathematical problems. Whilst working for the IBM, he iterated the '$z = z^2 + c$' equation, where he observed a bug-like structure repeating. This result famously became known as 'The Mandelbrot set', and this discovery founded modern fractal geometry. Mandelbrot's conception of fractals was to model nature in a way that captures roughness; this is a conception that goes against some of modern maths, like calculus which evolves around smooth shapes.

There are three techniques of generating fractals: escape time fractals (defined by a recurrence relation at each point in a space), iterated function fractals (which have a fixed geometric replacement rule) and random fractals (which are generated by sophisticated modelling). A Sierpinski triangle is an example of an iterated function fractal. It is an equilateral triangle subdivided recursively into smaller triangles. When originally looked at, you can see a large triangle split into three more identical triangles, which can go on forever.

Now before we go further into the mathematics of fractals, we need to understand what is meant by the term dimension. In mathematics, dimension is a measure of roughness; it is a ratio providing a statistical index of complexity comparing how detail in a pattern changes with the scale at which it is measured. We all know: a single point has a dimension of 0; a line segment has a dimension of 1; a square has a dimension of 2 and a cube has a dimension of 3. It is the norm to think that dimensions have to be an integer number! But, would you believe that a Sierpinski triangle has a dimension of approximately 1.585? It does seem a bit shocking, doesn't it? To help explain this, we will think of 3 non-fractals: a line segment, a square and a cube.

If we split a line segment, of length 1, into half, we are left with 2 line segments of ½. The scale factor of measure (length) becomes ½, which is equal to $(½)^1$.
Suppose we split a square of length 1, into segments of side length ½, we are left with 4 squares. The scale factor of measure (area) becomes ¼, which is equal to $(½)^2$.
Furthermore if we split the cube, of length 1, into segments of side length ½, we are left with 8 cubes. The scale factor of measure (volume) becomes $\frac{1}{8}$, which is equal to $(½)^3$.

Now do you notice something about the powers in the equivalent fractions for the measure scaling factor? They all represent the dimension of those shapes!

If we split the original Sierpinski triangle, of side length 1, into segments of side length ½, we get 3 triangles, hence the measure scale factor is $\frac{1}{3}$. Therefore we can imply that $(½)^d = \frac{1}{3}$.

We can now solve for d, to calculate the dimension of this triangle:

$$dlog\left(\frac{1}{2}\right) = \log\left(\frac{1}{3}\right)$$

$$d = \frac{\log\left(\frac{1}{3}\right)}{log\left(\frac{1}{2}\right)}$$

Therefore d=1.585 (to 3 decimal places). This means that a Sierpinski triangle is 1.585-dimensional. If you were to calculate the length of this pattern, it would be infinite. In addition if you were to calculate the area of this shape, you would get an answer converging towards zero. This may seem strange, but the fact that this shape is 1.585-dimensional explains this phenomenon.

However, how was it possible to find this link between fractals and their dimensions? This idea of fractal dimension all came to talk because of the British coastline! Surprising? Before the idea of fractals, in 1967, Mandelbrot was studying the relationship between changing the length of a ruler, and the measured length of the British coastline. Mandelbrot hypothesised that increasing measured lengths with smaller rulers is because coastlines have a property called self-similarity. This enabled Mandelbrot to first use 'fractal dimension', which helped find further dimensions about fractals such as the Sierpinski triangle. Mandelbrot's study about the British coastline observed that coastlines do not have a well-defined length, and therefore it can be hard to give a value to this, as coastlines can be said to be infinite. This phenomena is referred to as the 'coastline paradox'. Therefore it can be argued for coastlines in three dimensional space, a finite area can be contained by an infinite length of coastline. This may sound bewildering, but the coastline length rather than converging towards a number actually diverges infinitely. If you compare coastlines to fractals, it can be said that coastlines are produced by natural phenomena, hence compared to ideal fractals, which are made through iterative processes, they can be thought of less definite.

Fractals can also be found in nature, although they aren't mathematically exact. An example of a fractal can be seen in a snowflake. The crystallisation of water, which causes the formation of a snowflake, creates an iterative pattern. A Koch Snowflake can help you imagine why a snowflake can have a fractal behaviour. Physically, these snowflakes can be created by dividing a line segment into three segments of equal length. Then you draw an equilateral triangle that has the middle segment as its base and points outward. After this you remove the line segment that is the base of the triangle. If you repeat this many times you can notice a self-similar repeating pattern within the snowflake. Each iteration in this pattern increases the number of sides by a scale factor of 4. Again, like the coastline paradox, this pattern would have an infinite side length, but a finite area. We can also use this information to calculate the fractal dimension of this pattern, which would approximate to around 1.262 (to 3 decimal places).

The first use of fractals came right after Mandelbrot released his seminal work in 1975 on fractals. The first use was in developing computerised mountain ranges, which were developed solely on the mathematics of fractal dimensions. This occurred in 1978, with credit due to Loren Carpenter. The use of fractals now in the world around is quite significant. They can be used to stimulate bacterial growth; they can be used to create graphical effects in movies like Star Wars and they can be used in meteorology with cloud formation and air flow. This would not have been possible without understanding the mathematics of fractals, especially the fractal dimension component possessed by many self-similar shapes.

It is quite astonishing to think that this area of mathematics is unfamiliar to many people, despite it being found everywhere, from smaller structures, such as a fern; to gigantic structures, like black holes and the universe around. Even parts of our human body, for example the lungs, have fractal properties. Although we know the lungs have a specific volume, the true length of these specific organs are unknown, as there are disagreements on size, from some people saying it is the size of a garden, whilst others claim it is the size of 5 football pitches! We, mathematicians, know the correct answer is actually that the length of a lung is infinite! Fractal geometry is more than just art: it creates a new perspective in seeing the world differently from what was seen before, and this area of mathematics can help model the Earth in a more useful manner, which could help solve more complex problems.