Important Theorems in Euclidean Topology

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1 Introduction

The development of topology has been one of the most important milestones in mathematics of the 20th century. Its roots can be traced back to Euler and his 1736 paper on the *Seven Bridges of Königsberg* [4]. In this essay I aim to present four of the most influential theorems in Euclidean topology [16], namely the Borsuk-Ulam theorem, the Hairy Ball theorem, the Jordan Curve theorem, and the Brouwer Fixed Point theorem. Firstly their history will be mentioned, along with a discussion of their statements and applications. Then I shall provide a wider context for the understanding of the relationships between them, including many other theorems within the scope of topological progress of the last century.
This essay assumes some familiarity with basic concepts within topology, especially including homotopy theory.

2 The Borsuk-Ulam Theorem

The Borsuk-Ulam theorem was first referenced in Lyusternik and Shnirel'man (1930). It was first proved by Borsuk in 1933, who gave Ulam credit for the formulation of the problem in a footnote. [21, p. 25] The Borsuk-Ulam theorem states that for every continuous map from a 2-sphere into Euclidean 2-space, there exists a pair of antipodal points. [7, p. 133] Antipodal points are points on spheres that are diametrically opposite, so the straight line connecting them passes through the centre of the sphere. More formally:

> *Define a continuous map f: $S^2 \to \mathbb{R}^2$. Then there exist antipodal points w and -w in S^2 such that $f(w) = f(-w)$.*

The proof follows:
First we shall show that we need only concern B^2, instead of S^2. Define a map $g : S^2 \to \mathbb{R}^2$ by

$$g(w) = f(w) - f(-w) \qquad w \in S^2$$

We want to show that g must vanish at some point of S^2. We will only need to use the following property:

$$g(-w) = -g(w) \qquad w \in S^2$$

Consider the continuous map $g : B^2 \to \mathbb{R}^2$ defined by,

$$h(x, y) = g(x, y, \sqrt{1 - x^2 - y^2}) \qquad (x, y) \in B^2$$

According to this mapping, h is the result of flattening the top half of S^2 onto B^2. Using the aforementioned property we have,

$$h(-z) = -h(z) \qquad z \in S^1$$

The reason we specify $z \in \ S^1$ is that this is the unit circle, and so it the set of points in B^2 that contain antipodal points of the half-sphere. It is the equator of the half-sphere. It is therefore sufficient to show that any map h from B^2 to $\mathbb{R}^2$ satisfying the previous property vanishes at some point of B^2. We can construct a proof by contradiction. Suppose that an h does not vanish on B^2. We can construct another map:

$$\phi(z) = \frac{h(z)}{|h(z)|} \frac{|h(1)|}{h(1)} \qquad z \in B^2$$

defines a map with our desired properties. More specifically we have defined a map $\phi : B^2 \to S^2$ that satisfies,

$$\phi(-z) = -\phi(z) \qquad z \in S^1$$

$$\phi(1) = 1$$

Consider the path:

$$\alpha(s) = \phi(e^{2\pi i s}) \qquad 0 \leq s \leq 1$$

We can show that its index is zero.[1]. We can obtain a contradiction by showing that its index is odd.
Define a loop in B^2 by,

$$\beta(s) = e^{2\pi i s} \qquad 0 \leq s \leq 1$$

Then $\alpha = \phi(\beta)$. Since B^2 is a convex subset of $\mathbb{R}^2$, α is homotopic (with endpoints fixed) to the constant loop in S^1 at 1. In other words, α can be shrunk to the point 1 in S^1. We can then deduce that the have the same index, 0.

[1]Note that *index* is referred to by many authors as *degree* For more detailed reasoning as to why the index is zero, see the proof of Brouwer Fixed Point theorem

Set k : [0,1] $\rightarrow \mathbb{R}$ such that,

$$\phi(e^{2\pi i s}) = e^{2\pi i k(s)} \qquad 0 \leq s \leq 1$$

$$k(0) = 0$$

Then ind(α) = $k(1)$. Since ϕ is an odd function:

$$\exp[2\pi i k(s + \frac{1}{2})] = -\exp[2\pi i k(s)] = \exp[2\pi i (k(s) + \frac{1}{2}))] \qquad 0 \leq s \leq 1$$

For each fixed $s \in [0, \frac{1}{2}]$, the number

$$k(s + \frac{1}{2}) - k(s) - \frac{1}{2}$$

must be an integer. As the above number depends continuously on s and has a discrete range, it is constant. Set this constant value equal to an integer m, so:

$$k(s + \frac{1}{2}) - k(s) = m + \frac{1}{2} \qquad 0 \leq s \leq \frac{1}{2}$$

Then,

$$\text{ind}(\alpha) = k(1) = k(1) - k(\frac{1}{2}) + k(\frac{1}{2}) - k(0),$$

which gives an odd integer as shown by

$$m + \frac{1}{2} + m + \frac{1}{2} = 2m + 1$$

Thus we have obtained a contradiction, so h does vanish on B^2, and the theorem is proved. One interpretation of this theorem is that if you take a rubber ball, deflate it, crumple it and lay it flat, then there are two points that were antipodal on the surface of the ball that are now lying on top of each other. Another classic interpretation of the theorem is that at any time, there are two antipodal points on the surface of the Earth with equal temperature and, simultaneously, the same pressure. The single dimensional case is easier to prove. We can construct the odd function in an analogous way. As it is both positive and negative in the domain, it follows that the function vanishes in that domain by the Intermediate Value theorem, and the theorem is proved. Interestingly, the Borsuk-Ulam theorem has a combinatorial analog, that is, a combinatorical statement which is equivalent [23]. It is called Tucker's Lemma and this surprising link between topology and combinatorics is part of a field called topological combinatorics. An interesting implication of this theorem is the Ham Sandwich theorem, and its 2 dimension counterpart, the Pancake theorem. The Ham Sandwich theorem states [7, p. 134]:

Let U, V, and W be 3 bounded connected open subsets of $\mathbb{R}^3$. Then there is a plane in $\mathbb{R}^3$ that divides each of the sets into two pieces of equal volume.

The Pancake theorem states:

Let U and V be 2 bounded connected open subsets of $\mathbb{R}^2$. Then there is a straight line that divides each of U and V in half by area.

3 The Hairy Ball Theorem

The Hairy Ball theorem is the statement that there is no continuous field of non-zero tangent vectors on an even-dimensional n-sphere.[22]The theorem was first stated by Poincaré in the 19th century and was proved by Brouwer in 1912. When tangent vectors are visualised as hairs, the statement becomes 'you can't comb a hairy ball without creating a tuft'. Interestingly, the single-holed torus is the only compact, orientable surface (2 dimensional manifold) that has a non-vanishing tangent vector field![13] In fact, the Hairy Ball theorem can be thought of as a consequence of the remarkable Poincaré-Hopf theorem. This shows that the sum of the indices of a vector field on a compact, differentiable manifold over all of its (isolated) zeros is independent of which vector field is considered.[7, p. 188] Importantly, the number (which must be an integer) is equal to the Euler characteristic of the manifold. The most applicable consequence of this to the Hairy Ball theorem is that if you have a compact, differentiable manifold with a non-vanishing tangent vector field, its Euler characteristic must be zero.[27] Examples of manifolds with zero Euler charactertic are the torus, odd dimension n-spheres, the Möbius loop, and Klein bottle. All of these shapes have continuous non-vanishing tangent vector fields,(the latter 2 are non-orientable surfaces, so the torus is the only orientable surface with the desired property) which are fairly easy to imagine (apart from maybe the case of the Klein bottle).[14].
There are many proofs of this famous theorem, which were formulated at various points in time. Now I present a sketch of Milnor's analytic proof of the Hairy Ball theorem[22], as I believe that this is the easiest to follow without a foundation in homotopy theory.
Theorem 1:

An even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors.

The sphere S^{n-1} is trivially the set of all vectors $\mathbf{u} = (u_1, ...u_n)$ in Euclidean space $\mathbb{R}^n$ such that the Euclidean length $||\mathbf{u}|| = 1$. A vector $\mathbf{v}(\mathbf{u})$ in $\mathbb{R}^n$ is

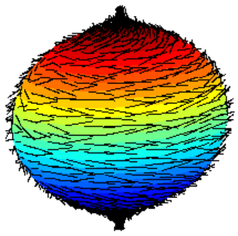


Figure 1: A failed attempt to comb a hairy ball [5]

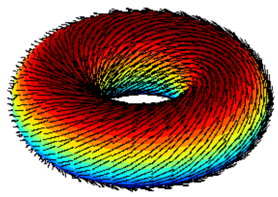


Figure 2: A successful attempt to comb a torus [6]

tangent to S^{n-1} at $\mathbf{u}$ if the Euclidean inner (dot) product $\mathbf{u} \cdot \mathbf{v}(\mathbf{u})$ equals zero. We can now show that if $n-1$ is odd, it is possible to define a differentiable field of unit tangent vectors on S^{n-1}, by

$$\mathbf{v}(u_1, ..., u_n) = (u_2, -u_1, ..., u_n, -u_{n-1})$$

Note that

$$(u_2, -u_1, ..., u_n, -u_{n-1}) \cdot (u_1, ...u_n) = u_2 u_1 - u_1 u_2 + ... + u_n u_{n-1} - u_{n-1} u_n = 0$$

This is why the Hairy Ball theorem is specified only for even-dimensional n-spheres; when n-1 is even, n is odd so in the above equation we would not be left with zero.
Let A be a compact region in $\mathbb{R}^n$ and let $\mathbf{x} \to \mathbf{v}(\mathbf{x})$ be a continuously differentiable vector field which is defined throughout a neighbourhood of A. For each real number t, consider

$$\boldsymbol{f_t}(\boldsymbol{x}) = \boldsymbol{x} + t\boldsymbol{v}(\boldsymbol{x})$$

which is defined for all x in A. Lemma 1:

> *If the parameter t is sufficiently small, then this mapping $\boldsymbol{f_t}$ is one-to-one and transforms the region A onto a nearby region $\boldsymbol{f_t}(\boldsymbol{A})$ whose volume can be expressed as a polynomial function of t.*

Proof:
Since A is compact, and the function $\mathbf{x} \to \mathbf{v}(\mathbf{x})$ is continuously differentiable, there exists a Lipschitz condition c such that [2]

$$||\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})|| \leq c||\boldsymbol{x} - \boldsymbol{y}||$$

[2] Proof of this claim can be found in [22]

Now choose any t such that $|t| < c^{-1}$. Then $\boldsymbol{f_t}$ is one-to-one. This is easy to see if we consider the implications of setting $\boldsymbol{f_t(x)} = \boldsymbol{f_t(y)}$. It would imply $\boldsymbol{x} - \boldsymbol{y} = t(\boldsymbol{v(y)} - \boldsymbol{v(x)})$, hence $||\boldsymbol{x} - \boldsymbol{y}|| \leq |t|c||\boldsymbol{x} - \boldsymbol{y}||$. We know that $|t|c < 1$, this implies that $\boldsymbol{x} = \boldsymbol{y}$.
The matrix of first derivatives (also known as the Jacobian matrix) of $\boldsymbol{f_t}$ can be written as $I + t[\frac{\partial v_i}{\partial x_j}]$, where I is the identity matrix [25]. Therefore its determinant must be of the form $1 + t\boldsymbol{\sigma_1(x)} + ... + t^n\boldsymbol{\sigma_n(x)}$, which is a polynomial in t with coefficients as continuous functions of $\boldsymbol{x}$. This determinant is strictly positive for $|t|$ sufficiently small, as there is always the t^0 constant term. To find the volume of an n-dimensional function $f(x_1, x_2, ...x_n)$, we integrate over its domain, D[15]:

$$\int \cdots \int_D f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

$$\text{Volume } \boldsymbol{f_t(A)} = \int \cdots \int_{f_t(A)} dV$$

$$= \int \cdots \int_A |\boldsymbol{J}|\, dx_1 \ldots dx_n$$

$$= \int \cdots \int_A \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} dx_1 \ldots dx_n$$

$$= \int \cdots \int_A \left| \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \end{bmatrix} + t \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial v_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{bmatrix} \right| dx_1 \ldots dx_n$$

$$= \int \cdots \int_A \left| \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} + t \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial v_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{bmatrix} \right| dx_1 \ldots dx_n$$

Therefore

$$\text{Volume } \boldsymbol{f_t(A)} = \int \cdots \int_A 1 + t\boldsymbol{\sigma_1(x)} + ... + t^n\boldsymbol{\sigma_n(x)}\, dx_1 \ldots dx_n$$

$$= a_0 + a_1 t + ... + a_n t^n$$

with coefficients

$$a_k = \int \cdots \int_A \boldsymbol{\sigma}_k\boldsymbol{(x)}\, dx_1 \ldots dx_n$$

The lemma is now proven.
Lemma 2:

If the parameter t is sufficiently small, then the transformation $\boldsymbol{u} \to \boldsymbol{u} + t\boldsymbol{v}(\boldsymbol{u})$ *maps the unit sphere in* $\mathbf{R}^n$ *onto the sphere of radius* $\sqrt{1+t^2}$.

This seems obvious enough, as if $||\boldsymbol{u}|| = ||\boldsymbol{v}(\boldsymbol{u})|| = 1$, then $||u + tv(u)|| = \sqrt{1+t^2}$. For a more formal proof, consult [22].
We can now prove Theorem 1. As region A we take the region between two concentric spheres, such that $a \leq ||\boldsymbol{x}|| \leq b$. We extend the vector field $\boldsymbol{v}$ throughout this region by setting $\boldsymbol{v}(r\boldsymbol{u}) = r\boldsymbol{v}(\boldsymbol{u})$ for $a \leq r \leq b$, so $\boldsymbol{f_t}(r\boldsymbol{u}) = r\boldsymbol{f_t}(\boldsymbol{u})$. It follows that the mapping $\boldsymbol{f_t}(\boldsymbol{x}) = \boldsymbol{x} + t\boldsymbol{v}(\boldsymbol{x})$ is defined throughout the region A, and maps the sphere of radius r onto the sphere of radius $r\sqrt{1+t^2}$, if t is sufficiently small. This is because we now have the radius as $\sqrt{r^2 + r^2t^2}$ instead of $\sqrt{1+t^2}$. Hence $\boldsymbol{f_t}(\boldsymbol{x})$ maps A onto the region between spheres of radius $a\sqrt{1+t^2}$ and $b\sqrt{1+t^2}$. Since the volume of an $(n-1)$-sphere is proportional to r^n[20],

$$\text{Volume } \boldsymbol{f}_t(A) = (\sqrt{(1+t^2)})^n \text{ Volume(A)}$$

Thus, if n is odd, this volume is not a polynomial function of t. Comparing this with Lemma 1, we have a contradiction, and Theorem 1 is proved. Theorem 1 is very similar, but not identical to the statement of the Hairy Ball theorem. The proof that it is implied used the Weierstrass Approximation theorem and is very short. Again, it can be found in [22].

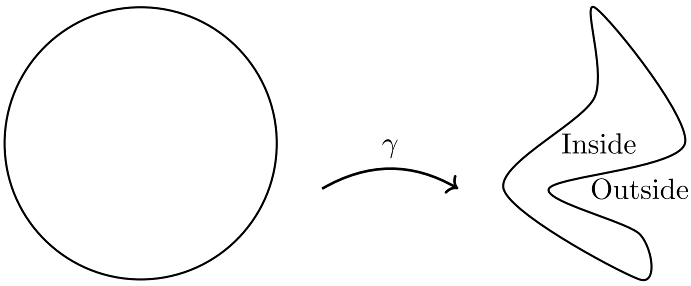
4 The Jordan Curve theorem

The Jordan Curve theorem is the statement that any continuous loop in the plane (that is non self-intersecting) divides the plane into two regions: an inside and an outside. This statement is intuitively true, it is obvious to any child that you have to cross a line to get from the outside to the inside of the circle. In fact, the very words 'inside' and 'outside' imply some sort of disconnection. Despite the obvious nature of the statement, it is difficult to prove. This may not become such a surprise when one considers that this must hold true for *any* Jordan curve. It is not hard to imagine that the theorem may be difficult to prove in the case of nowhere-differentiable curves such as fractal curves (for example the Koch Snowflake or the Osgood curve). Despite the The Jordan Curve theorem being widely known, many professional mathematicians have not read a proof of it.[30]
Historically, there seems to be some uncertainty about the origin of the first valid proof. The theorem was first stated by Camille Jordan in 1893 in his book *Cours d'Analyse*[12]. His proof was subsequently widely criticised, and most authors cite the first correct proof to be due to O. Velben in 1905. Recently, there has been controversy as to the validity of Jordan's proof. Thomas Hales argues that Jordan's proof is essentially correct, and

quotes Michael Reeken: "Jordan's proof is essentially correct... Jordan's proof does not present the details in a satisfactory way. But the idea is right and with some polishing the proof would be impeccable."[9] Here is the formal statement of the theorem:
A Jordan Curve in $\mathbb{R}^2$ is the image of a one-to-one continuous mapping γ of S^1 into $\mathbb{R}^2$, denoted by Γ. The mapping is a homeomorphism onto its image,

$$\Gamma = \gamma(S^1)$$

The Jordan Curve theorem states that $\mathbb{R}^2 \backslash \Gamma$ is disconnected and consists of two components. Alternatively it states that $\mathbb{R}^2 \backslash \Gamma$ has precisely two connected components (i.e. both components are they themselves connected), which is an equivalent statement. Here is a diagram illustrating the theorem:



Due to the length of the proof required, I shall not provide it here. I point the reader to [30] for a rigorous proof of the statement.

5 The Brouwer Fixed Point Theorem

The Brouwer Fixed Point theorem is a well known fixed point theorem in topology. It was first proved in 1910 by Hadamard and Brouwer, and has seen many applications since. In economics it played a vital role in the work that gained Kenneth Arrow and Gérard Debreu the 1950 Nobel prize. It has even has applications in game theory, where John Nash used it to prove that there is a winning strategy for white in the game of Hex. In its most general form, it states:[3]

> Given that a set $K \subset \mathbb{R}^n$ is compact and convex, and that a function $f : K \to K$ is continuous, then there exists some $c \in K$ such that $f(c) = c$.

This means that there is some fixed point c that is left unchanged by the map. One way to illustrate this is that if you take a map of the world and lay it out on a table, there will always be a "you are here" point on the map. Another way of visualising it is to imagine stirring a drink in a glass.

When the liquid comes to rest, there will always be a point in the liquid that ends up in the same place that it started. The following examples show the importance of the pre-conditions on K; that K is compact (thus closed and bounded) and convex.

Boundedness:

Take

$$f(x) = x - 1$$

which is a map from $\mathbb{R}$ onto itself. $\mathbb{R}$ is convex and closed, but unbounded. No point is mapped to itself, which is evident as the line is parallel to $f(x) = x$. Therefore, we must specify that functions are bounded for the Brouwer Fixed Point theorem.

Closedness:

Take

$$f(x) = x^2$$

which is a continuous function from the open interval $(0, 1)$ is to itself. $(0, 1)$ is convex and bounded, but not closed. It also does not intersect $f(x) = x$ in the domain, so no point is mapped to itself. Therefore, we must specify that functions are closed.

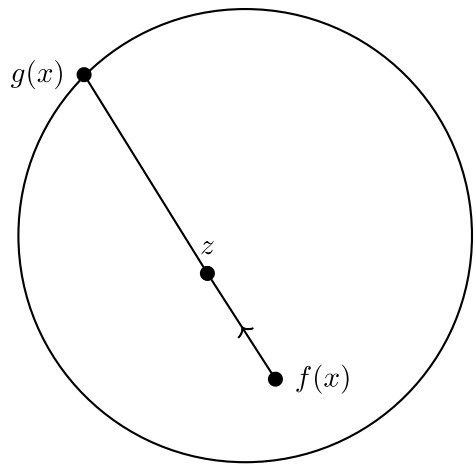
Convexness:

Convexness is not strictly necessary for Brouwer's Fixed Point theorem as it holds for domains which are homeomorphic to the unit ball B^n. Therefore the domain must be simply connected, but is not necessarily convex. For this reason, the theorem does not hold for domains with holes, as they aren't simply connected. If you imagine a circular racetrack, it is obviously not simply connected as it is made from a larger circle with a smaller one removed. A car on the racetrack can always move forward around the track, so continuous map can be defined without any fixed points. If you shrunk the radius of the removed circle, eventually the racetrack would become B^2 and the centre would be a fixed point. A generalisation of Brouwer's Fixed Point theorem for 'hole-less' domains follows from the Lefschetz Fixed Point theorem, which is another fixed point theorem that generalises Brouwer's theorem.

Proving a special case

Any map $B^2 \to B^2$ has a fixed point.

We shall construct a proof by contradiction. For each $z \in B^2$, let $g(z)$ be the point of S^1 at which the ray from $f(z)$ passing through z leaves B^2. Then g is a continuous map from B^2 to S^1. This is intuitive but proof can be found elsewhere [7, p.223]. Clearly $g(z) = z$ if $z \in S^1$.



Lemma 1:

> *Let f be a map from B^2 to S^1 that satisfies $f(1) = 1$. Then the loop α, defined by*
>
> $$\alpha(s) = f(e^{2\pi is}) \qquad 0 \leq s \leq 1$$
>
> *has index zero:*
>
> $$ind(\alpha) = 0$$

Before I proceed with the proof, I shall state the following without proof, which can be found here [7, p. 122].

> Let Y be a convex subset of $\mathbb{R}^n$, let $y_0, y_1 \in$Y, and let f be a map from Y to X. If α and β are paths from y_0 to y_1, then $f \circ \alpha$ is homotopic to $f \circ \beta$ with endpoints fixed.

Proof: Define a loop β in B^2 by

$$\beta(s) = e^{2\pi is} \qquad 0 \leq s \leq 1$$

Then $\alpha = f \circ \beta$. Since B^2 is convex, the above shows that α is homotopic with endpoints fixed to the constant loop in S^1 at 1. Since they are homotopic, they have the same index. The constant loop makes no turns around the circle, so its index is 0, as is α's. ind(α)=0.
Lemma 2:

> There is no map of B^2 onto S^1 such that $f(z) = z$ for all $z \in S^1$.

Proof:
Assume the map, f, exists. Then the loop $s \to e^{2\pi i s}, 0 \leq s \leq 1$, should have index zero, by lemma 1. But the loop winds around the circle exactly once, and so has index 1[3]. As we have obtained a contradiction, the lemma is proved.
Now back to our original proof. Since $g(z) = z$ if $z \in S^1$, we obtain a contradiction to lemma 2, and this special case of the Brouwer Fixed Point theorem is proved.

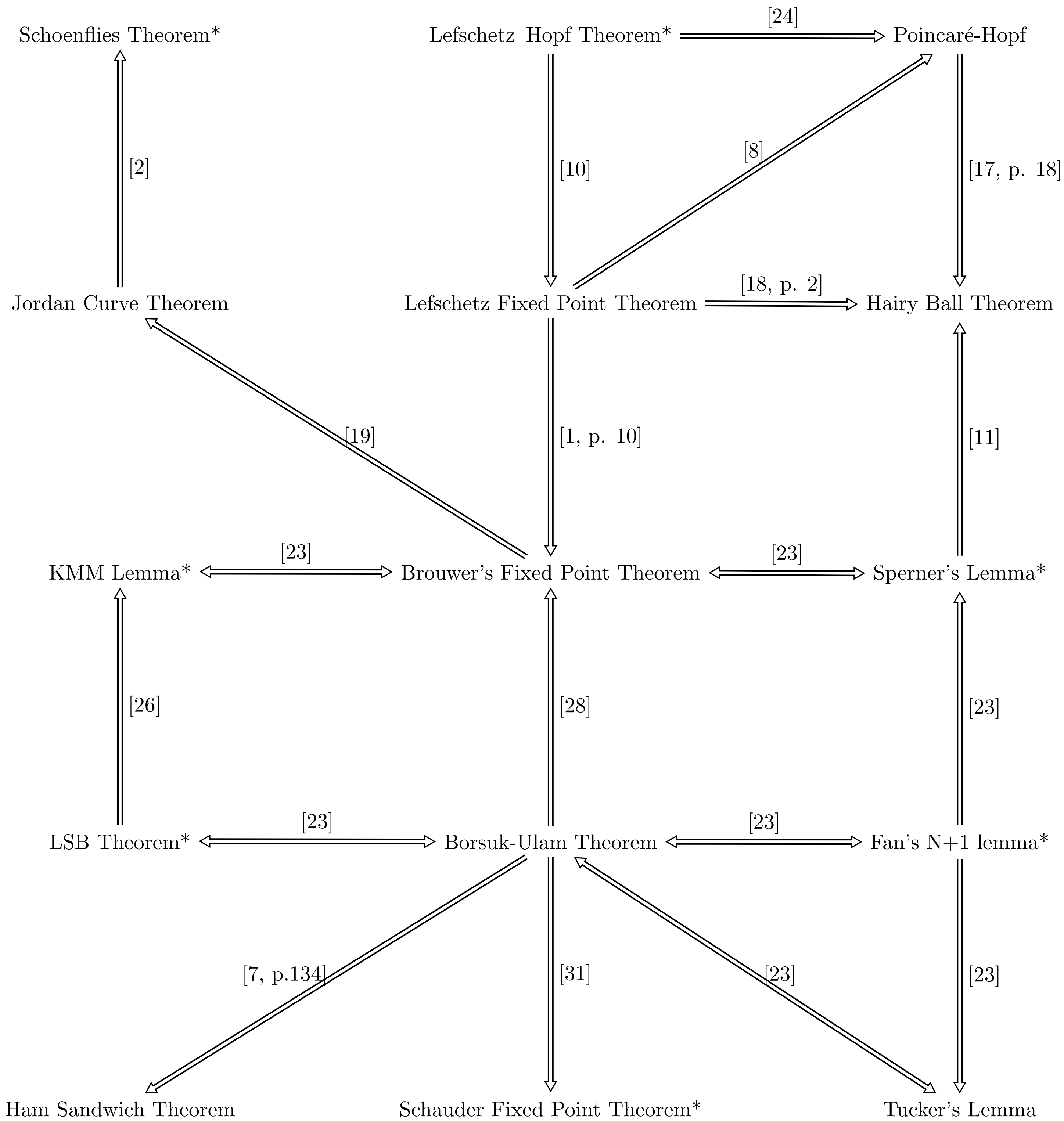
6 Equations in Context

So far I have presented a few of the most well known theorems in Euclidean topology. This still begs the question: how are they all connected? To gain a bigger picture , I have constructed a map using various sources. An arrow from one theorem to another means that the latter is *implied* by the former, or a direct proof of the latter can be constructed by the former. The motivation behind this map may be unclear as, in formal logic, any true statement implies any other [7, p. 167]. Despite this, it can still be useful to create maps of this kind (showing direct implications), as can be seen in [23] and [29]. The purpose is twofold: superficially it shows which statements can be deduced directly from which others, but its primary objective is to give a sense of connection between separate ideas. Starred theorems are not mentioned in this essay, and the reader may wish to consult the sources provided. The diagram in question can be found on the next page.

7 Conclusion

I have now presented some of the key theorems in Euclidean topology. Hopefully, if nothing else, this essay has clearly demonstrated the interlinked and collaborative nature of mathematical progress. All of the theorems mentioned were built upon pre-existing ideas developed throughout the careers of the mathematicians before them. For example, many of the concepts involved in this essay were born Poincaré's 1895 paper *Analysis Situs*, and have been developed over the last century. I shall end by quoting Newton: "If I have seen further than others, it is by standing upon the shoulders of giants".

[3]Note that in this situation, the index represents the number of full turns the loop makes around the circle.


Schoenflies Theorem*
Lefschetz–Hopf Theorem*
[24]
Poincaré-Hopf
[2]
[10]
[8]
[17, p. 18]
Jordan Curve Theorem
Lefschetz Fixed Point Theorem
[18, p. 2]
Hairy Ball Theorem
[19]
[1, p. 10]
[11]
KMM Lemma*
[23]
Brouwer's Fixed Point Theorem
[23]
Sperner's Lemma*
[26]
[28]
[23]
LSB Theorem*
[23]
Borsuk-Ulam Theorem
[23]
Fan's N+1 lemma*
[7, p.134]
[31]
[23]
[23]
Ham Sandwich Theorem
Schauder Fixed Point Theorem*
Tucker's Lemma

References

[1] D. Becker. “The Lefschetz fixed point theorem”. https://www.math.leidenuniv.nl/scripties/BachBecker.pdf. July 2016.

[2] R. H. Bing. “The Geometric Topology of 3-Manifolds”. In: vol. 40. Colloquium Publications, 1983, pp. 29–32.

[3] Colin Buxton. “Brouwer Fixed-Point Theorem”. http://math.stmarys-ca.edu/wp-content/uploads/2017/07/Colin-Buxton.pdf. May 2016.

[4] Fred Croom. “Principles of Topology”. In: Cengage Learning, 2008, p. 7.

[5] Public Domain. “A failed attempt to comb a hairy ball flat.” In: *Wikimedia Commons* (Mar. 2007).

[6] Public Domain. “Combing a hairy doughnut flat.” In: *Wikimedia Commons* (Mar. 2007).

[7] T. Gamelin and R. Greene. *Introduction To Topology*. 2nd ed. Dover, 1999.

[8] Ariel Hafftka. “Differential Topology and the Poincaré-Hopf Theorem”. http://www.math.uchicago.edu/ may/VIGRE/VIGRE2009/REUPapers/Hafftka.pdf.

[9] Thomas C. Hales. “Jordan’s Proof of the Jordan Curve Theorem”. In: *Studies in Logic, Grammar and Rhetoric* 10 (2007).

[10] I. M. James. “History of Topology”. In: North Holland, 1999, pp. 271–299.

[11] Tyler Jarvis and James Tanton. “The Hairy Ball Theorem via Sperner’s Lemma”. In: *The American Mathematical Monthly* 111.7 (2004), pp. 599–603. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/4145162`.

[12] Camille Jordan. “Cours d’Analyse de École Polytechnique”. In: vol. 3. Paris: Gauthier-Villars, 1887, pp. 587–594.

[13] Donald W. Kahn. “Introduction to Global Analysis”. In: Dover, 1980. Chap. 4, p. 96.

[14] Cameron Krulewski and Jenny Walsh. “The Hopf Degree Theorem: Homotopy Groups and Vector Fields”. https://math.uchicago.edu/ may/REU 2017/Cameron2.pdf.

[15] Ron Larson and Bruce Edwards. *Multivariable Calculus*. 10th ed. Cengage Learning, 2014.

[16] Daniel Leborgne. “Calcul différentiel et géométrie”. In: PUF, 1982, p. 15.

[17] Jonathan Libgober. "The Euler Characteristic, Poincare-Hopf Theorem, and Applications". http://www.math.uchicago.edu/ may/VIGRE/VIGRE2010/REUPapers/Libgober.pdf.

[18] Edgar Lin. "An Overview and Proof of The Lefschetz Fixed-Point Theorem". http://math.uchicago.edu/ may/REU2019/REUPapers/Lin,Edgar.pdf.

[19] Ryuji Maehara. "The Jordan Curve Theorem Via the Brouwer Fixed Point Theorem". In: *The American Mathematical Monthly* 91.10 (1984), pp. 641–643. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2323369`.

[20] NIST Digital Library of Mathematical Functions. "Gamma Function Applications, equation 5.19(iii)". https://dlmf.nist.gov/5.19.

[21] Jiří Matoušek. *Using the Borsuk-Ulam Theorem.* Springer, 2008.

[22] John Milnor. "Analytic Proofs of the "Hairy Ball Theorem" and the Brouwer Fixed Point Theorem". In: *The American Mathematical Monthly* 85.7 (1978), pp. 521–524. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2320860`.

[23] Kathryn L. Nyman and Francis Edward Su. "A Borsuk–Ulam Equivalent that Directly Implies Sperner's Lemma". In: *The American Mathematical Monthly* 120.4 (2013), pp. 346–354. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/10.4169/amer.math.monthly.120.04.346`.

[24] Paul Renteln. "Manifolds, Tensors and Forms". In: Cambridge University Press, 2014, p. 260.

[25] Walter Rudin. "Principles of Mathematical Analysis". In: 3rd ed. McGraw-Hill, 1976, p. 234.

[26] Gwen Spencer and Francis Edward Su. "The LSB Theorem Implies the KKM Lemma". In: *The American Mathematical Monthly* 114.2 (2007), pp. 156–159. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/27642146`.

[27] Norman Steenrod. "The Topology of Fibre Bundles". In: Princeton University Press, 1951. Chap. 3, p. 203.

[28] Francis Edward Su. "Borsuk-Ulam Implies Brouwer: A Direct Construction". In: *The American Mathematical Monthly* 104.9 (1997), pp. 855–859. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2975293`.

[29] Tianyu Sun and Wensheng Yu. "Formalization of the Axiom of Choice and its Equivalent Theorems". In: *CoRR* abs/1906.03930 (2019). arXiv: `1906.03930`. URL: `http://arxiv.org/abs/1906.03930`.

[30] Helge Tverberg. "A Proof of the Jordan Curve Theorem". In: *Bulletin of the London Mathematical Society* 12.1 (1980), pp. 34–38. DOI: 10.1112/blms/12.1.34. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/blms/12.1.34. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/blms/12.1.34.

[31] Alexey Yu. Volovikov. "Borsuk-Ulam Implies Brouwer: A Direct Construction Revisited". In: *The American Mathematical Monthly* 115.6 (2008), pp. 553–556. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/27642536.