Euler's Identity

In my school, there is a tree with the inscription $e^{i\pi} = -1$.

I, of course, recognised this as Euler's identity. I had read about it before, and I knew how it's often viewed as "The Most Beautiful Theorem in Mathematics" because it brings together three really powerful constants: e, i and π. However, it was unclear to me how these three could possibly be linked in such an identity as Euler's, so I decided to investigate its origin.

I began with Maclaurin Expansions, which express a polynomial function in terms of a sum of terms based off of its derivatives.

To work towards a general formula, I started by defining any polynomial in terms of a and x.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...$

$f(0) = a_0$

I then took the first derivative of the polynomial above, to get:

$f' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + ...$

$\therefore f'(0) = a_1$

Next, I took the second derivative:

$f'' = 2a_2 + 3(2)a_3x + 4(3)a_4x^2 + 5(4)a_5x^3 + ...$

$\therefore f''(0) = 2a_2$

I noticed that:

$$a_2 = \frac{f''(0)}{(2)(1)} = \frac{f''(0)}{2!}$$

Finally, I took the third derivative:

$$f''' = 3(2)a_3 + 4(3)(2)a_4x + 5(4)(3)a_5x^2 + \ldots$$

$$\therefore f'''(0) = 3(2)a_3$$

Similarly, I noticed that:

$$a_3 = \frac{f'''(0)}{(3)(2)(1)} = \frac{f'''(0)}{3!}$$

This led me to the general formula:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^r(0)x^r}{r!} + \ldots$$

To work towards Euler's identity, I studied the Maclaurin Expansion of e.

In Maths, e is a famous constant, and irrational number, approximately 2.71828.

e has the unique property such that it is the only function for which the nth derivative of e^x will always be e^x, i.e. $f^r(x) = f(x)$.

Therefore, I worked out the Maclaurin expansion of e by:

$$Let\, f(x) = e^x$$

$$f(0) = e^0 = 1$$

$$f'(x) = e^x$$

$$\therefore f'(0) = e^0 = 1$$

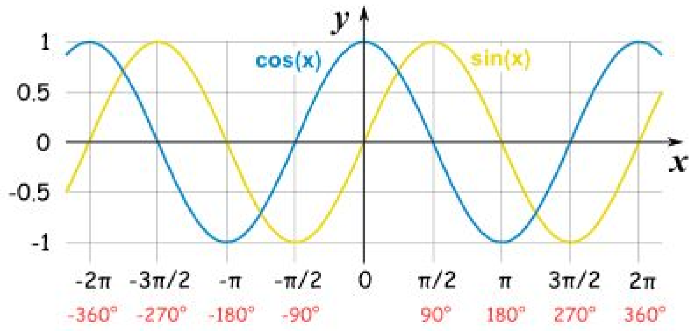
$$f''(x) = e^x$$

$$\therefore f''(0) = e^0 = 1$$

$$f^r(x) = e^x \Rightarrow f^r(0) = e^0 = 1$$

$$\therefore\ e \equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Next, I looked at the Maclaurin expansions for the trigonometric functions $sin(x)$ and $cos(x)$, whose graphs are shown below.



$Let\ f(x) = sin(x)$

$f(0) = sin(0) = 0$

I know that the first derivative of $sin(x)$ is $cos(x)$, and I wanted to understand why, so I observed the function $sin(x)$ first. I noticed at the points where the curve crosses the x-axis, the gradient is at its steepest, either in the negative or positive direction. I also noticed that in the peaks and troughs of $sin(x)$, the gradient of the curve is 0.

Therefore, $f'(x) = 0$ when $f(x)$ is at its maximum points and minimum points, and $f'(x)$ will have its maximum and minimum points where $f(x) = 0$.

When I plotted it, the curve that I got looked very much like $cos(x)$, and therefore the derivative of $sin(x)$ is $cos(x)$.

$f'(x) = cos(x) \quad \therefore f'(0) = 1$

$f''(x) = -sin(x) \quad \therefore f''(0) = 0$

$f'''(x) = -cos(x) \quad \therefore f'''(0) = -1$

$f^{iv}(x) = sin(x) \quad \therefore f^{iv}(0) = 0$

From this, I noticed a recurring pattern: $[0, 1, 0, -1]$ and so on.

This is clearly because of the periodic properties of $sin(x)$, where it repeats exactly every 2π degrees.

Using this, I found that:

$sin(x) = 0 + x + 0(\frac{x^2}{2!}) - (\frac{x^3}{3!}) + 0(\frac{x^4}{4!}) + (\frac{x^5}{5!}) - ...$

I then generalised and simplified this to get my final equation:

$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^z x^{2z-1}}{(2z+1)!}$

By following the same process, I found that $cos(x)$ had the same recurring pattern of results from the n^th derivative as $sin(x)$, just starting at a different point: $[1, 0, -1, 0]$ and so on. This is because $cos(x)$ also has periodic properties, however, it is shifted π degrees away from $sin(x)$ so the two graphs are offset. It follows that the recurring sequence of $cos(x)$ would start halfway through the sequence for $sin(x)$.

Therefore, the final equation that I reached for $cos(x)$ was:

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... + \frac{(-1)^z x^{2z}}{(2z)!}$$

I then used these three derived formulae to work forwards, so I have stated them again here.

1. $e = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

   I noticed that this expansion contains all terms of x, and all of the terms are positive.

2. $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^z x^{2z-1}}{(2z+1)!}$

   This expansion contains only the odd powers of x, and there are oscillating signs.

3. $cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... + \frac{(-1)^z x^{2z}}{(2z)!}$

   This expansion contains only the even powers of x, and there are again oscillating signs.

Next, I took equation 1, the expansion of e^x, and I substituted ix for x.

i is an imaginary number which represents $\sqrt{-1}$.

To do this I had to look at the values of i to the n^th power, as follows.

- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$

Much like $sin(x)$ and $cos(x)$, there seems to be a recurring pattern of results. This is very useful for infinite series because it allows us to generalise.

When I substituted e^{ix} for e^x, I got the following result:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$e^{ix} = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right]$$

When the expansion of e^x is factorised, as shown above, the two parts resemble equations 2 and 3: the expansions of $sin(x)$ and $cos(x)$.

$\therefore\ e^{ix} = cos(x) + i \cdot sin(x)$

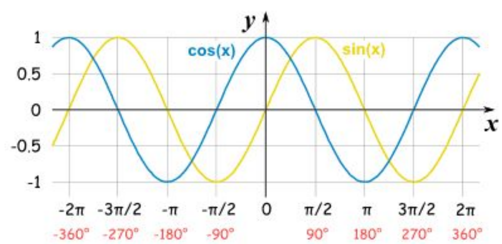
For my next step, I tried to substitute π for x.

$e^{i\pi} = cos(\pi) + i \cdot sin(\pi)$

From studying the trigonometric graphs of $sin(x)$ and $cos(x)$, shown again below, I found the following values:

- $cos(\pi) = -1$
- $sin(\pi) = 0$

  $\therefore\ i \cdot sin(x) = 0$



Therefore, I could rewrite the above equation as:

$e^{i\pi} = -1 + 0$

$e^{i\pi} = -1$

This is Euler's identity, and hence my investigation is complete.