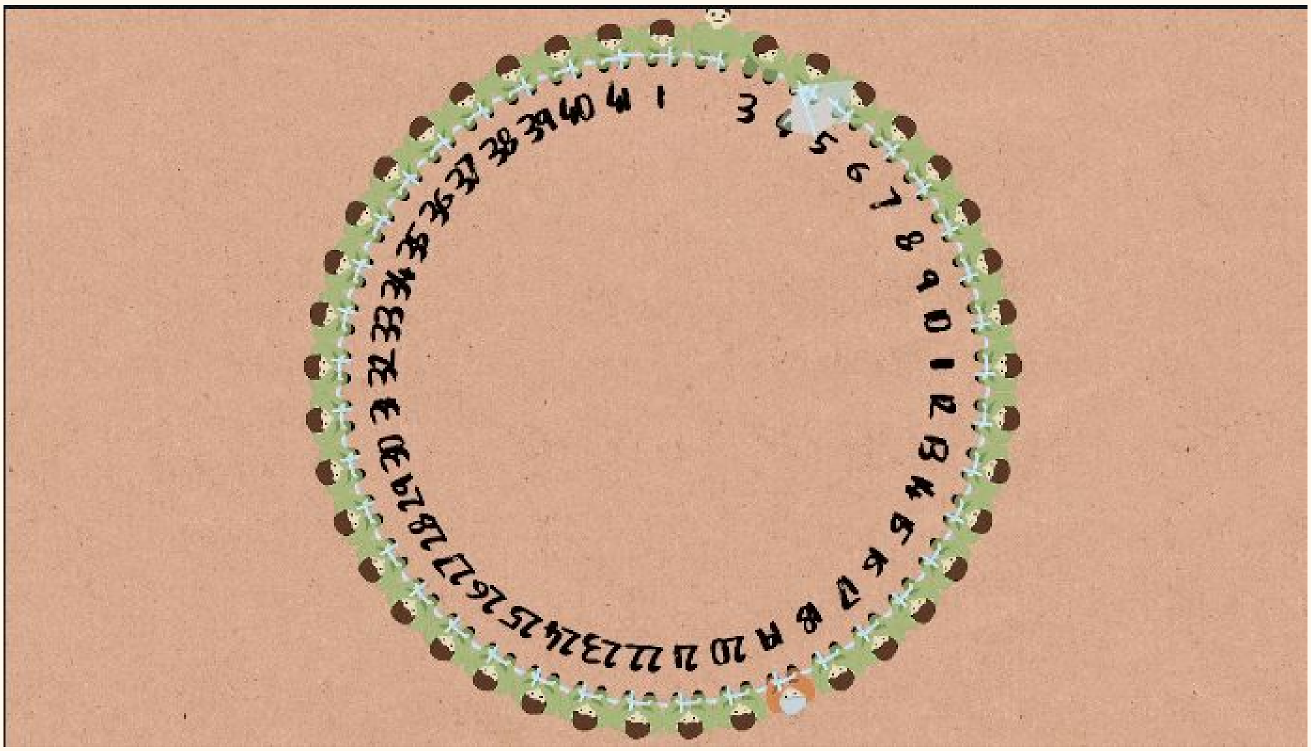
TEDDY ROCKS MATHS ESSAY COMPETITION
2020

COMBINATORICS
THE JOSEPHUS PROBLEM

—

By Jaimy Sajit



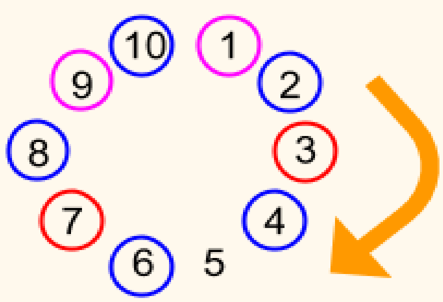
What exactly links a card game, a Romano-Jewish historian and computers together? This is an easy one to wake you up - binary, of course! This essay intends to show the path taken to reach this conclusion, as well as my own worked solution to the problem and some other titbits along the way.

There existed, in the first century, a historian from Roman Judea by the name of Flavius Josephus. This was a time of war and *The Great Revolt*, of which Josephus played the part of a revolter. The story goes that Josephus and forty other rebel Jews were trapped in a cave by a number of well-armed Roman soldiers. Rather than be captured, they found more solace in dying at each other's hands and so devised a sort of suicide pact that would keep them their pride, if not their life:

they decided to stand in a circle and, in turn, kill the nearest man who was alive to their leftside. This would continue until there was all but one man left, who would kill himself.

A simple task, yet Josephus wanted out: he would rather be captured by the Romans than die. It seemed there was a problem - yet Josephus had a solution. If all the men killed each other one by one, the last man would be commiting suicide - but not necessarily, for if this man was Josephus himself then he could freely hand himself over to the Roman soldiers and escape a ghastly reunion with his brothers in arms. So the question, arguably worth more than a million dollars, was thus: where should Josephus stand in this circle, so he was the sole survivor?

We can phrase this problem mathematically, with a general number of rebel Jews, *n*, numbered around the circle from 1 to *n*. The first pass around the circle proves it is clearly not beneficial to stand in an even numbered space, since these are all killed first. Then the number of rebels halves and the same is repeated. One can experiment with different values of *n*, to find the survivor's position in each case.



The diagram to the left shows the scenario for *n*=10.
The positions executed on the **first**, **second** and **third** passes around the circle, with position 1 initiating are indicated.
The person originally in position number five would be the survivor here.

The execution list is:
1→ **2**, 3 → **4**, 5 → **6**, 7 → **8**, 9 → **10**, 1 → **3**, 5 → **7**, 9 → **1**, 5 → **9**

Table 1 shows some values of *n* and their corresponding survivor positions, *s(n)*.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
s(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Table 1

A quick glance over the table will reveal that, as we concluded from intuition, all the winning positions are odd numbers. A pattern emerges of the odd numbers increasing by two, then resetting to 1 at certain numbers. The observant reader will notice these numbers have the distinct property of being pure powers of 2 (2^0, 2^1, 2^2, 2^3 and 2^4 shown in Table 1).

We can represent all values of *n* as

$$n = 2^x + m, \text{ where } x \geq 0. \qquad \mathbf{1}$$

n	2^0+0	2^1+0	2^1+1	2^2+0	2^2+1	2^2+2	2^2+3	2^3+0	2^3+1	2^3+2	2^3+3	2^3+4	2^3+5	2^3+6	2^3+7	2^4+0
$s(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Table 2

From Table 2, it is important to notice the relationships between the value of *m* and *s(n)*.

m	0	1	2	3	4	5	6	7
$s(n)$	1	3	5	7	9	11	13	15

So for the general value of *m,*

$$s(n) = 2m + 1 \qquad \mathbf{2}$$

Rearranging **1** gives

$$m = n - 2^x$$

Which can be substituted into **2** giving

$$s(n) = 2n - 2^{x+1} + 1$$

The original problem then, of n = 41,
yields a solution of

$$s(41) = 2(41) - 2^{(5+1)} + 1$$
$$= 19$$

Another way to think about the relationship between *m* and *s(n)* is that after *m* number of executions, it is the turn of *s(n)* to execute. For example, in the case of n = 10 = $2^3 + 2$: after the second person is executed, it is the turn of person 5, who turns out to be *s(n)* in this scenario.

This gives that $s(n) = 2m + 1$ as before.

Therefore $s(41) = 2(9)+1$
$= 19$

So Josephus would have to stand in the 19th position to keep his life.

The relations to the powers of 2 have not ended, however, as a striking shortcut comes about when *n* is written in binary.

E.g when $n = 41$, $n = 2^5 + 2^3 + 2^0$

Therefore the binary number is 101001

A trick that I found while researching the subject is that moving the first digit to the end gives the value of *s(n)* as a binary expansion, such as shown here 010011

Which gives $s(n) = 19$ as found earlier.

It is worth noting that the action of shifting all the numbers to the left is the same as multiplying all the digits by 2, since we are working in powers of 2 , and adding the leading number 1 to the end, which is the equivalent of the second algebraic method found, of $s(n) = 2m + 1$.

The conclusion to Josephus' tale is not quite as perfect as we would have hoped, since it turned out he had an accomplice that made their chance of survival even greater. At least *we* haven't cheated.

A similar idea is used in a card game called *Out and Under* where cards are alternately moved to the bottom of the pack or discarded. This would be the Josephus problem with parameters of k=2, where k refers to the k^{th} person executed.

What drew me to this problem was not only its interesting context but also how simple and 'neat' the solution was considering the exercise seemed so complex, which is often the case with mathematics. I first stumbled across this problem while binge-watching Numberphile videos, and while watching the video I admired how we continuously built on observations that complement the data, producing useful equations that led us to the solution. This type of problem-solving is very different from the more methodical approach I have used throughout my time at school, where we learn the method before we see the problem. Here we are inventing our own methods, and as shown, different interpretations can lead to different outcomes (in this case, equations). The unexpected way the binary representation was used to find a shortcut is also intriguing.

This sort of problem could fall under the umbrella of combinatorics, involving counting and arranging finite structures. This branch of maths lends itself to clear-cut solutions and methods which make hours of working redundant and our lives much simpler!

In any case, when I am trapped in a cave and face a similar situation, at least I'll know where to stand - and so do you. Let's hope we aren't in there together!