Group Theory, Quarks and Country Dancing

Groups are everywhere – in music, poetry, art and science. They are not only an incredibly abstract, intricate, and complex area of maths, but are also an amazingly simple and often beautiful concept. From quarks to country dancing, they have many real life applications, and over the next few paragraphs I intend to explore some of these, and show how important group theory is to everyday life.

In its simplest terms, group theory is the study of symmetry, or more technically, the study of transformations which have some level of invariance. A group is a set of elements or actions which can undergo a binary operation, following certain axioms (conditions on which an abstract concept is based).

The first axiom is that a group must be closed, meaning any elements of the group that undergo the binary operation, must produce a result that is another element of the group. This means that for m and n, where m and n are two elements of the group, not necessarily distinct, $m*n = p$, where * denotes a binary operation (an operation involving two variables) and p is another element of the group.

The second axiom states that the binary operation must be associative, meaning that $a*(b*c) = (a*b)*c$, where a, b and c are all elements of the group.

The third axiom states that the group must have an identity, which is to say that there is an element of the group, let's call it e, such that $n*e = n$ where n represents any other element of the group.

The fourth and final axiom states that every element of the group must have an inverse, meaning that for every element in the group, which we shall call n, there is another element (which could be n again), which we shall call m, where $n*m = e$, where e is the identity.

Now that we understand what a group refers to, we can begin to explore its applications. In geometry, group theory can be used to look at the transformations of shapes, the simplest example being an equilateral triangle. If you let the elements of the group be the three rotations (0°, 120° and 240°) and the three reflections (in each line going through one corner of the triangle and bisecting the opposite side), and if you allow the binary operation to be a combination of two of these transformations, you can produce a group, which follows all 4 axioms stated above. This simple group, is one of the most recognisable to abstract mathematicians and is one of only two groups containing 6 elements. The following table, known as a Cayley table, shows this group. In the table, I refers to 'do nothing' (or rotate 0°), R1 and R2 refer to rotations of 120° and 240°, and X, Y and Z refer to the three reflections:

	I	**R1**	**R2**	**X**	**Y**	**Z**
I	I	R1	R2	X	Y	Z
R1	R1	R2	I	Z	X	Y
R2	R2	I	R1	Y	Z	X
X	X	Y	Z	I	R1	R2
Y	Y	Z	X	R2	I	R1
Z	Z	X	Y	R1	R2	I

In this group, I is the identity, and each element has an inverse as follows:
I, X, Y and Z are all self inverse (performing the operation on the element with itself produces the identity) and R1 and R2 are each others' inverses.

One of the more unusual uses of group theory is in the study of quantum physics. Everything in the world, that we know, is made up of atoms, and atoms are made up of protons, neutrons and electrons. In 1964, the quark model was proposed which suggested protons and neutrons were made up of even smaller fundamental particles, called quarks. The three most common quarks found are the up, down and strange quark. These quarks make up hadrons, which are simply a mix of 2 or 3 quarks, which are called mesons and baryons respectively. Protons and neutrons are both baryons, protons are made from two up quarks and one down quark, and neutrons are made from one up quark and two down quarks. Strange quarks also make up other hadrons, however these quickly decay and so are not often found.

If you imagine a cube, you can see it has 24 symmetries, or rather, that it has 24 orientations in which it can be placed, and still look the same, whilst actually being in a different orientation. For example, if you place a cube on the table, and then rotate it 90°, you would have found two of it's possible 24 symmetries. These symmetries can be put into a group, where the elements include transformations such as 'do nothing', 'reflect in y-axis' and 'rotate 90° clockwise around z-axis'.

Hadrons are exactly the same as this, and have symmetries in almost the same way as the cube does. As up, down and strange quarks are all very similar, having very similar properties and roughly equal weights. As a result of this, we find that we can create a group of baryons and a group of mesons. The only difference between hadrons and the cube is that as the quarks are only very similar, and not identical, only certain arrangements of them will work, and be perfectly symmetrical. As a result, scientists found that there were 8 baryons and 8 mesons which could be produced from these quarks. Of the 8 baryons produced, only 7 had been discovered so far, and thus, one new baryon had been predicted. This new baryon consisted of three strange quarks, and was named the omega minus. In 1964, the omega minus Particle was discovered, confirming that group theory was able to predict the existence of a new particle.

This breakthrough was huge for abstract maths, as it showed that even abstract mathematical ideas, which seemed to serve no purpose, were actually hugely useful, and when used in the correct way, could make groundbreaking scientific discoveries.

A more common place where group theory can be found is in music. There are many areas of music where fruit theory can be found, however in this example I am going to focus on pitch. In music there are 12 notes, or pitch classes, which make up each octave. We can imagine an octave to be a loop, starting at A, and then returning back to A, and we can assign each note with a number, A is assigned 1, A# is assigned 2 and so on, up to G# being assigned 12. We can then consider some transformations, for example T1, which refers to a shift one note clockwise, and T2, which refers to a shift two notes clockwise, or in general, Tn, which refers to a shift n notes clockwise. These shifts can then become the elements of the group, and not only that, but they each refer to the note with the same number assigned to it. For example, T2 refers to A#, which was assigned the number 2. We now find we have created a group, with elements A – G#, where each note represents a transformation as well as a note. Therefore, we can find we are able to create a group out of music, which can be seen represented by the Cayley table below:

	A	A#	B	C	C#	D	D#	E	F	F#	G	G#
A	A#	B	C	C#	D	D#	E	F	F#	G	G#	A
A#	B	C	C#	D	D#	E	F	F#	G	G#	A	A#
B	C	C#	D	D#	E	F	F#	G	G#	A	A#	B
C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
C#	D	D#	E	F	F#	G	G#	A	A#	B	C	C#
D	D#	E	F	F#	G	G#	A	A#	B	C	C#	D
D#	E	F	F#	G	G#	A	A#	B	C	C#	D	D#
E	F	F#	G	G#	A	A#	B	C	C#	D	D#	E
F	F#	G	G#	A	A#	B	C	C#	D	D#	E	F
F#	G	G#	A	A#	B	C	C#	D	D#	E	F	F#
G	G#	A	A#	B	C	C#	D	D#	E	F	F#	G
G#	A	A#	B	C	C#	D	D#	E	F	F#	G	G#

In this group, the identity is G#, and each element has an inverse as follows:
Within their pairs, A and G, A# and F#, B and F, C and E, and C# and D# are all each others' inverses, and G# and D are both self inverse.

A final area where group theory can be seen is in country dancing. In country dancing, a common arrangement is to have four people stood in a square arrangement, and then someone calls out instructions, and the four people follow, with the aim of returning to their original start position. Some examples of common moves include opposite people swapping place, everyone rotating position 90° clockwise, and opposite pairs swapping place. Such transformations are actually able to form a group, which turns out to be the same as the group for the transformations of a square, as this is essentially what it is, a square being transformed in many ways, with its vertices being represented by four people. It can therefore be said that group theory underlies country dancing, from the music that is danced to, to the moves that are made, showing abstract maths can appear in everyday life, and can have functions in the arts as well as in science.

Overall, from science to music to dance, we have seen that group theory can be applied to a wide variety of situations, and can be extremely useful in helping to explain ideas and develop new theories and explanations for patterns that we see. It's presence in everyday life is undeniable, and it's influence on the world of abstract maths is huge.

Citations:

The Use of Group Theory in Particle Physics – Sam Mayo
https://www.youtube.com/watch?v=NOMUnMuxDZY

The Framework of Music Theory as Represented With Groups – Ada Zhang
https://sites.math.washington.edu/~morrow/336_09/papers/Ada.pdf

Visual Group Theory – Nathan C. Carter
http://www.jstor.org/stable/10.4169/j.ctt5hh9nr.8