The Josephus Problem

The Siege of Yodfat took place over the span of forty seven days in 67 CE involving Roman troops who stormed Yodfat, a Jewish town, during the Great revolt, when Jewish civilians staged three major rebellions against the Roman Empire. The fighting took place in Roman-controlled Judea and consequently, the Roman forces overwhelmingly gained victory, prompting Jewish towns to be razed, the relocation of its people and the annexing of land for military purposes. During the siege, one Jewish soldier, Flavius Josephus, reported that he and his forty soldiers were imprisoned in a cave by Roman soldiers. Rather than give in to submission and be taken captive, they opted for death and proposed a method in which randomly, lots would be drawn, which would determine in what order the soldiers would be killed, and whom by. At the end, there would be just one person left and they were expected to commit suicide. Supposedly, Josephus was unkeen on the prospect of suicide but felt that if he revealed this to the rest of his men, they would feel betrayed and murder him anyway. Therefore, he sought to be the last person in his troop in order to surrender to the Romans and willingly face capture.

The group devised a system for their deaths, which involved all the men standing in a circle facing each other with their swords. The first person that they assigned would fatally stab the adjacent man to his left. The next surviving man in the circle, adjacent to the murdered man, would then kill the man to *his* left. This cycle would continue and once the first round was completed, the process would resume with a smaller group. This method would proceed until one soldier remained, with the plan being that they would commit suicide. Josephus and another man were alive until the end and they mutually agreed to succumb to Roman capture. Josephus claimed he did not know whether this feat was by luck or by the hand of God; however, this savagery has provoked a notion as to finding a formula to calculate the position of the last man standing, in a situation with any given number of people.

Here, the Josephus Problem is examined in the setting of a game of paintball involving any number of individual players.

For example, if a group started with 8 people in a circle, person 1 hits person 2 with a paintball, person 3 would hit person 4, person 5 will hit person 6 and person 7 is going to hit person 8 in the first round. In the second round, 1 hits 3 and 5 hits 7, and in the final round, person 1 hits 5, so person 1 wins.

If there is a group of 4 people, 1 hits 2, 3 hits 4 and 1 hits 3, so person 1 wins again. If there are just 2 people, 1 hits 2 and wins. The first significant trend in these scenarios, is that no one in a position of an even number wins. This is logical, since 1 will always start and hit 2, so the next remaining person is 3 who will hit 4, and so on, thus people in an even numbered position will be hit with a paintball. Another trend that could be considered at this stage is that person 1 will win. However, for 5 people, number 1 hits 2, number 3 hits 4, but number 5 hits 1, eliminating this prospect. Instead, person 3 will be victorious this time. Similarly, for 7 people, person 7 eliminates person 1 and in fact will go on to be the person who is not paint-balled and win. If the first 12 winners are displayed in a table, it would look like this:

Number of paintballers	Winner
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9

The general pattern that can be observed from this is that the winner is always odd and it increases by 2 each time up until a point where it resets and starts from 1 again. However, another observation is that the numbers in which the winner is person 1, are all pure powers of 2, including when there is 1 person as 2^0, is 1. If there are 16 paintballers, the same is true.

Any number can be made by adding 2^n (where n is any number) to another number, which can also be expressed as powers of 2. For example 85 can be written as 2 to the power of 6 (2^6) plus 2 to the power of 4 (2^4) plus 2^2 plus 2^0. If 64 + 21 is written as $2^a + j$, with a being the greatest power of two beneath the number of people (in this case, 85, so a is 6) and j being the sum of the rest of the numbers of two to the power of something (so $2^4 + 2^2 + 2^0 = 21$, so j is 21), then j can tell us which person will win. So the twenty-first non-hit person, will end up winning, which would be the forty-third person, as all the even people are hits eliminating them.

For a smaller example, consider a group consisting of 13 people.
The smallest power of two less than 13 is 8, 2^3.

$13 = 2^3 + 2^2 + 2^0$ (i.e: 8 + 4 + 1 = 13)

13 - 8 = 5

$5 = 2^2 + 2^0$ (i.e: 5 = 4 + 1)

So the fifth person who has not been hit, or the person immediately after 5 people have been hit, will be the victor. In this scenario, 1 hits 2, 3 hits 4, 5 hits 6, 7 hits 8, 9 hits 10 and so five hits have been made, meaning person 11 will win. Continuing the method, 11 hits 12, 13 hits 1, 3 hits 5, 7 hits 9, 11 hits 13 in the next round, leaving persons 3, 7 and 11. Finally, 3 hits 7 and 11 hits 3, culminating with 11 as the triumphant winner.

Therefore, overall the solution to the Josephus problem is using $2^a + j$, where a is the greatest power of 2 which is less than n, (the number of people in the group), and j is the difference between n and 2^a, which will enable the position of the last surviving paintballer to be determined by finding the position of the person immediately after j number of hits.

The Josephus problem does not have many practical applications, with being involved in game theory its main role in applied mathematics. However, its larger and more ingenious purpose is to explore different concepts of maths, not necessarily related, and attempt to interconnect them and utilise this combination to prove that there is a solution to the problem and showcase the simple sophistication of maths. It makes maths seem challenging yet rewarding and satisfying in the sense that it requires logic and thought, but also knowledge and awareness, to recognise the patterns, and in doing so captivates many people.