Is $e^{i\pi}+1=0$ really that amazing?

Everyone in the world of maths will have come across the famous equation " $e^{i\pi}+1=0$ ", and had the sense that this is beautiful: an equation combining the iconic numbers $0, 1, e, i$ and π.

The equation was printed in large writing on the corridor in our new mathematics department for all to see a year ago. We have a largely blank wall, and so this equation does stand out really well, and throughout the year, many of my pupils have asked me about it.

And here's the controversial bit: I'm not that enamored by it. There, I said it!

And to clarify, I am someone who regularly claims mathematics is an art form, and love the beauty in the subject and the amazing things to be found, but for me, this is just a specific case of Euler's Formula $e^{i\theta}=\cos(\theta)+isin(\theta)$. Perhaps as exciting as (to over exaggerate a little), the equation 1+2=3, oh look, an equation which has the first three positive integers all in a row, amazing!

However, I do enjoy investigating off-syllabus mathematics, and am known for saying things like "let's go on a mathematical excursion" or "just derive it". So to live up to my catch-phrases, I will attempt to share here how I might talk through all the mathematics leading up to the famous $e^{i\pi}+1=0$ …

So off we go: as stated, the equation appears as a special case of $e^{i\theta}=\cos(\theta)+isin(\theta)$, specifically when $\theta=\pi$. The left hand side of the equation is clear, and the missing pieces are that $\cos(\pi)=-1$ and $isin(\pi)=0$, leading to $e^{i\pi}=-1$, which can be rearranged. But how does that even work? Taking the cosine or sine of pi? Isn't pi the ratio of the diameter and circumference of a circle? And that's before we even think about what "i" and "e" even mean? Let's try and break it down a bit…

i, the imaginary number

Imaginary numbers? That's just silly, surely! Well, no, the acceptance of imaginary numbers into mathematics in the 16th Century has proven its worth over the years – in fact, their use in electronics is so crucial, that I suspect I wouldn't be sat here writing this essay on a computer without them. In the 16th century, Italian mathematicians used to duel each other in the streets, challenging each other to solve mathematical equations to see who was the best. As quadratics have a formula, they preferred to challenge each other to solve more challenging cubics or quartics. One such mathematician, Cardano, had found a formula, like the quadratic formula, for a cubic in a certain format, the so called 'depressed cubic': $x^3+ax+b=0$. The formula is[1]:

$$x=\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}+\sqrt[3]{-\frac{b}{2}-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}$$

[1] I'm not convinced spending a page deriving this formula is helpful, but if you're interested, a quick google found this 30min long video working through it! https://www.youtube.com/watch?v=n96jXP2zGEg

The issue is, that, for known solutions, the formula requires a brief foray into uncharted waters by square rooting a negative number. This does feel rather impossible, which is where the 'imaginary' part comes in, which in turn lead to the concept of complex numbers. Complex numbers have a real part, and an imaginary part, such as $2 + 3i$ (where the "2" is the real part and the "$3i$" is the imaginary part). The definition for i itself is that it is equal to $\sqrt{-1}$ and therefore $i^2 = -1$. This now allows the square root any negative number, for example $\sqrt{-4} = 2i$. This opens up a whole new world of opportunity, and has proven to be invaluable over the centuries.

e, Euler's number

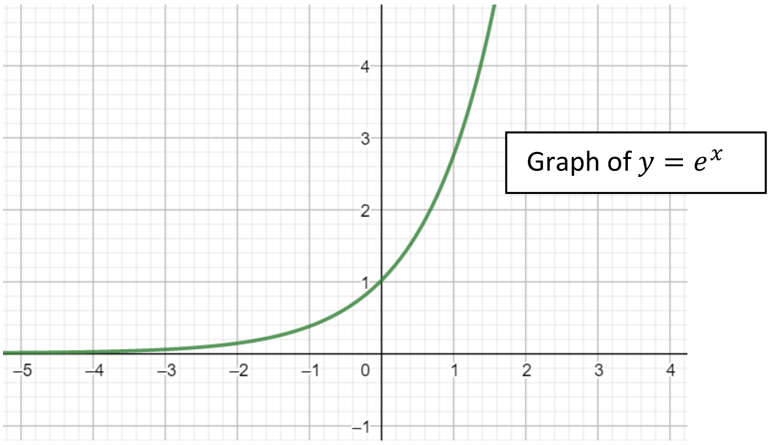
e, just like π, is an irrational number that features in a lot of mathematics, although unlike pi, there is no geometrical representation. So where does it come from? Euler was the first to really get to grips with it, and although there are many ways to think about e, the following is a nice starting point:

Consider you have £1 in the bank, and you have 100% annual interest. After 1 year, you would have £2. Now imagine that you split that 100% interest into two increases of 50%, every 6 months. So after 6 months you would have £1.50, and then at the end of the year, you would have 50% of that value, so a total of £2.25, an improvement. The same happens if you split the interest in 4 and calculate it at 4 times during the year. Below is a table to show the gains when you split the interest into smaller parts and more intervals during the year:

When the interest is split every:	Amount of money at the end of the year (rounded to 5 d.p.)
Year	£2
Half year	£2.25
Quarter	£2.44141
Month	£2.61304
Week	£2.69260
Day	£2.71457
Hour	£2.71813
Minute	£2.71828
Second	£2.71828

In fact, if you could have continuous compound interest, then you would gain the maximum of 2.718281828459… which you can see appearing in the tables. This infinite decimal is called e.

Another key aspect of e, is looking at the exponential graph of $y = e^x$ as e speaks the natural language of growth, whether financial or otherwise.

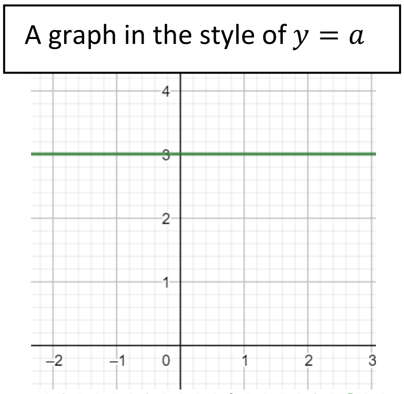


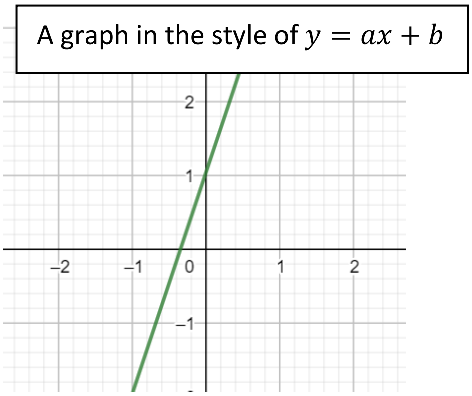
When talking about growth and rates of change, in mathematics, calculus has a crucial role to play, which involves differentiating and integrating functions. What makes $y = e^x$ so special, (and incredibly useful!) is the fact that when differentiated, it equals the same! That is to say, $\frac{dy}{dx} = e^x$ as well[2]. This is a unique function for which this works.

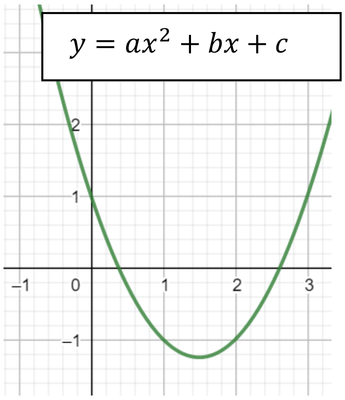
Off to infinity

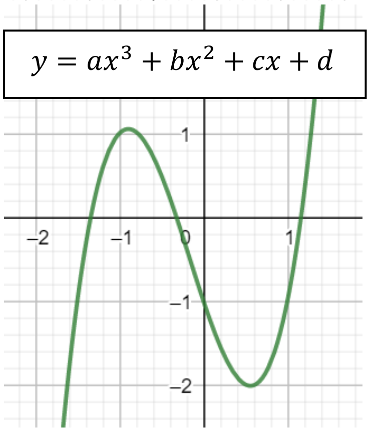
This next bit is arguably one of the most difficult to understand, and so part of this may get a bit 'hand wavy' for the benefit of grasping the concept in order to move on. When we consider polynomials of increasing order, it is possible to visualize how each additional power has the potential to create another bend in the graph. The higher the exponent is of x, the more 'bendy' potential the graph has. Here are some examples of polynomials with increasing powers of x to show this:

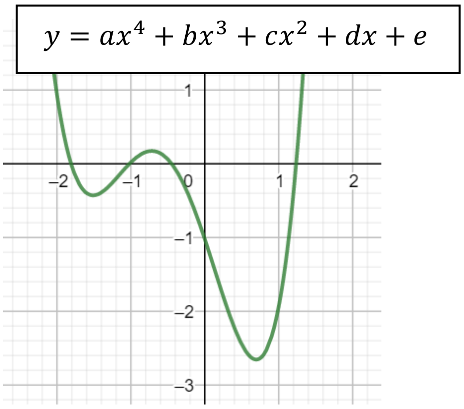
[2] Again, proving this through differentiation from first principles is quite a long process when considering this article, but by all means google around, there are plenty of websites and videos which can convince you if you are interested.


A graph in the style of $y = a$


A graph in the style of $y = ax + b$


$y = ax^2 + bx + c$


$y = ax^3 + bx^2 + cx + d$


$y = ax^4 + bx^3 + cx^2 + dx + e$

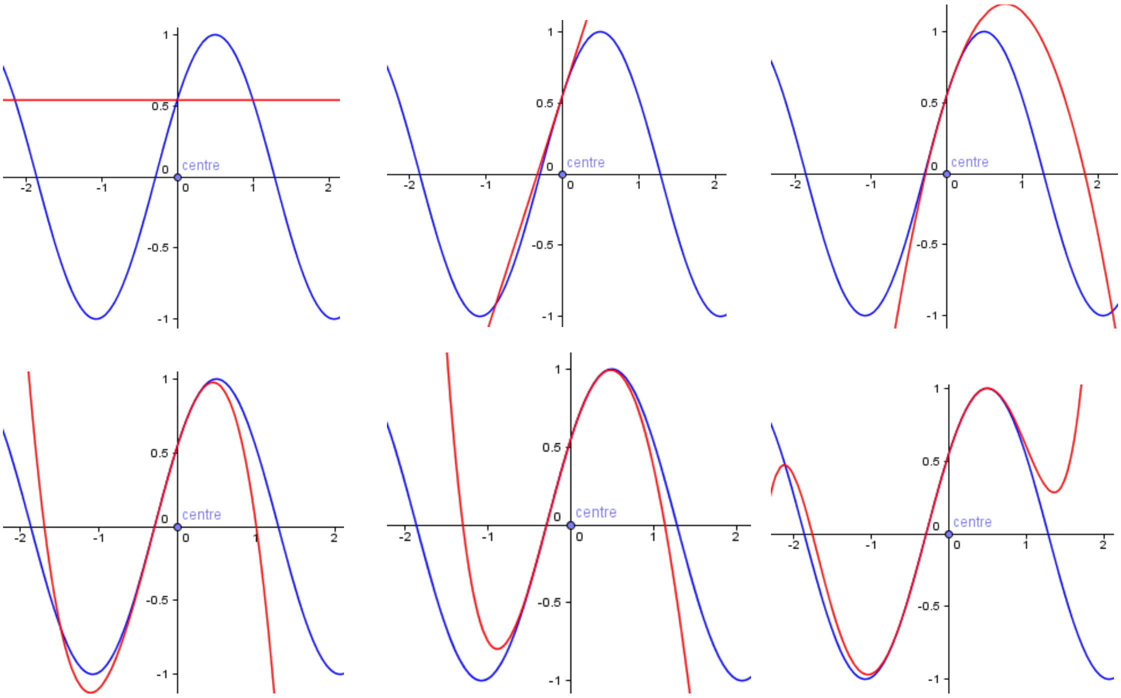
This would suggest that with clever manipulation, one could use a polynomial to approximate any graph, such as $y = \cos(x)$. As one might suspect, the more terms you add, the better the approximation; with the idea that if you could sum infinitely many powers of x together, you would actually replicate $y = \cos(x)$ perfectly. That is an incredibly powerful tool, that a graph, or more specifically, a function, has the potential to just be written as an (infinite) sum of powers of x. A polynomial is potentially easier to deal with, for example, when solving equations or easily obtaining good enough approximations for real life use in physics. The infinite summation formulae are called Taylor series, and they rely on some conditions, such as that the function you are approximating can keep on differentiating. Thinking back to $y = e^x$, and it's differential, also e^x, then this is exactly one such function. The Taylor Series starts at a specific point, and slowly builds up the infinite polynomial by considering differentiation[3]. When starting at $x = 0$, then this case is often referred to as a Maclaurin Series, with the following formula:

$$f(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 + \cdots$$

It looks scary, but it's not too bad, the 'dashes' which turn into numbers denote the amount of times you differentiate the function, so for example, $f^4(x)$ means you have differentiated the $f(x)$ function 4 times. And the 'exclamation marks' behind the numbers are also a significant short-hand. It means "factorial", and for example, $4!$ means $4 \times 3 \times 2 \times 1$. The second term above in theory has a "$1!$" in the denominator, but that is just "1", and for the first term, you could technically have a "$0!$", but by definition, that is also just equal to 1.

[3] Once again, the derivation of the proof of the Taylor series is rather long, but this video should be helpful for anyone interested: https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-11/v/maclaurin-and-taylor-series-intuition

When using the infinite formula, you can choose how many terms you will include, depending on the accuracy required. Below are some graphs where the blue function is being approximated by the red one. In each step, the red function involves more and more terms of the infinite series, which visually adds more "bends" and therefore allows for a better approximation:



As you can see, already with 3 or 4 terms the approximation is actually really accurate around the starting point of $x = 0$. [4]

Using the Taylor series, we can now try and calculate the series for the functions $y = e^x$, $y = \cos(x)$, and $y = \sin(x)$. The e^x function is not too difficult, as to find "$f(0)$", you compute $e^0 = 1$, and no matter how many times you differentiate e^x, you just keep having e^x. That means all $f^n(0) = 1$, making the expansion of $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

And what's nice here, is if you differentiate the right hand side of the above equation, you end up just getting the same again, which you should, as we have said e^x doesn't change when differentiated!

$y = \cos(x)$ and $y = \sin(x)$ are slightly more tricky, because you need to know how to differentiate them. Again, unfortunately I will have to provide spoilers at this stage and reveal that if you differentiate $\cos(x)$ you end up with $-\sin(x)$. And if you differentiate $y = \sin(x)$, you end up with $\cos(x)$. [5] And so there is a cyclic nature to both trigonometric functions if you keep on differentiating. This means we can use the Taylor series to obtain:

[4] To clarify, you don't have to start at $x = 0$. The Taylor series lets you start at any value you want, making it an excellent tool to approximate a particular value or area of a function you are interested in without having to add too many terms.

[5] This can again be shown with differentiation from first principles, although it requires knowledge of radians which we haven't covered, yet! You could try and think about what the gradient of $y = \cos(x)$ is at a number of points (eg. When $x = 0°$, the gradient is 0 or when $x = 90°$, the gradient is -1) and try and see how those values change to convince yourself somewhat of the results.

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Even to the untrained eye I think these have a certain beauty in their patterns, and patterns are such a key idea when investigating mathematics. In fact, looking at them a bit more, it feels like combining the $\cos(x)$ and $\sin(x)$ expansion could somehow match the e^x expansion. Well, for this to work, we need to dip back into the imaginary world and make use of i, and replace x with ix in the expansions. You will need to think carefully how powers of i work in order to obtain the result.[6] Armed with all this information, and hopefully not too much of a headache, it is possible to arrive at Euler's equation: $e^{ix} = \cos(x) + isin(x)$. Now to resolve that $\theta = \pi$ part from earlier…

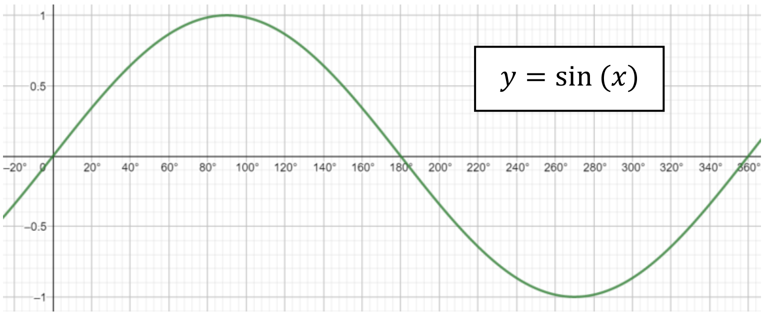
Radians, the natural way to measure angles

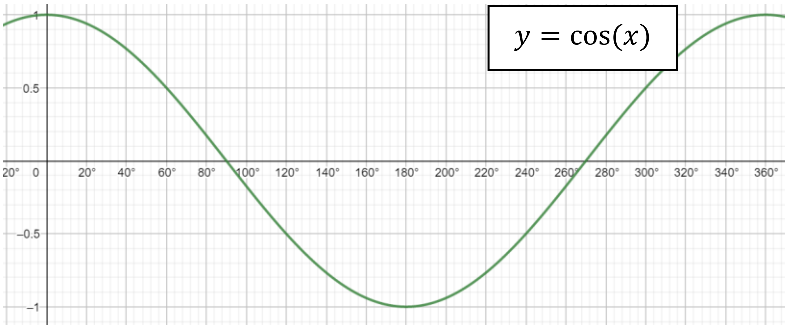
Radians, like degrees, are a unit for measuring angles with. For degrees, there are 360 degrees in a full circle, why? Well, in some ways, "why not" – you have to split the circle somehow if you want to count units of angle measurement. The history books suggest it may have been to do with either the sexagesimal number system of the ancient Babylonians, or either that the year roughly has 360 days. If you are thinking that the sexagesimal (base 60 rather than our base 10) is bizarre, then remember that we have 60 seconds in a minute, and 60 minutes in an hour. The number 360 does have an excellent amount of factors, making it easy to split a cake between a large variety of numbers of friends.

What about radians though? Well, let's think briefly about the definition of an angle: the size of an angle between two lines is measured by the amount one line has been turned in relation to the other. If those two lines happen to both be the radius of a circle, then the distance turned is the length along the circumference of the circle. If the distance travelled along the circumference happens to be the same distance as the radius, then this is 1 radian $\approx 57.3°$. And if you know the circumference of a circle, then it's no surprise that $2\pi\ radians = 360°$. So what's the big deal? Well, as radians are a length, it makes a whole range of things possible, such as calculating $\sin(x) + x$. If you tried to calculate that in degrees, you would be trying to add a 'normal' number to a degree number, which just doesn't work! It would be like trying to combine cm with kg. There are various formulae which rely on the angle being calculated in radians to work, and a significant one for this essay is their use in differentiating $\sin(x)$ and $\cos(x)$ from first principles, which is not possible with degrees. Radians may seem alien initially, but they are the more 'natural' unit for angle measurement.

Now that we have all our ingredients, the final touch is to substitute $x = \pi$ into the formula. As mentioned, $\cos(\pi) = -1$ and $\sin(\pi) = 0$ which is quite visible from the graph of the trigonometric functions below, remembering that $\pi = 180°$ (I've put the x-axis in degrees to make it feel more familier!):

[6] Big hint: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and $i^5 = i$, which is back to where we started, which is another nice pattern.





And there you have it: $e^{i\pi} + 1 = 0$

Considering all the fascinating mathematics 'behind the scenes', has my opinion changed about $e^{i\pi} + 1 = 0$? Not really. If being really harsh, it's just saying that if you start at co-ordinates $(1,0)$ and turn $180°$ about the origin, you will end up at $(-1,0)$, which seems fairly trivial…

What *is* amazing though, is the journey: the maths encountered along the way is so ground-breaking and fundamental to modern mathematics, and that's only when considering the route I took. There are in fact a number of ways of arriving at the formula, which in itself is rather exciting. The implications and use of this formula are also astounding, from the pure mathematical aspect of considering what i^i equals (final spoiler: it's a real number!!), as well as numerous applied mathematics topics such as electronics and aerodynamics.

In conclusion, the equation itself is but a mere snap-shot in time, but is built upon a rich and amazing history of mathematical evolution, and has led to even further progress and has been fundamental to so many aspects of our modern mathematical world. And if the aesthetics of having $e, i, 1, 0$ and π all in one equation is able to make more people appreciate and love mathematics, then even I might concede, yes, it's an amazing equation!