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COLORING MAPS AND THE FOUR COLOR THEOREM
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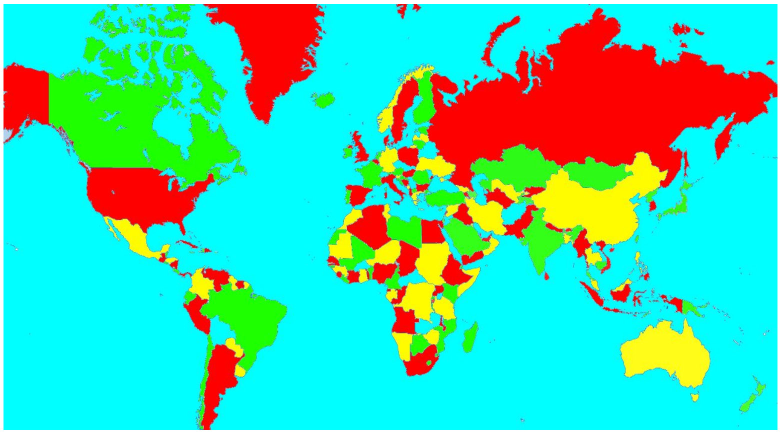
Observe the following world map[1] *very* closely:



Is there anything particularly intriguing about this map? Is it any different than what you might see in a history class? Anything special?

You might have uttered *no* to all of these questions. It is just a world map; it doesn't even have the names of the countries. There's nothing special to it. The only thing off about it is the use of neon colors capable of blinding you.

*Then why am I being asked this question? Surely there is something special that I'm not seeing, or I would not have been asked this.*

Unfortunately, you are correct. "You see, but you do not observe," as Sherlock Holmes once so eloquently said. If you look carefully, you'll notice how each country does not sit adjacent to another country or body of water that is the same color. Take Brazil, for example, which is colored in green. No country sits adjacent to Brazil that is also green. Pick any country or body of water and you'll notice the exact same thing.

*But what's extraordinary about that? I can take any set of crayons and do the same thing!*

Look *closer*. How many colors are there?

Just four. Red, yellow, green and blue.

Could you do the same thing with just three colors? Two? *Can you always color anything without having two adjacent regions identically colored with only four colors? Is there a case where you just can't use four? Will it ever be necessary to color five?*

This peculiar question stumped thousands of mathematicians worldwide for over a century until it was proved by Kenneth Appel and Wolfgang Haken in 1976.

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1 https://upload.wikimedia.org/wikipedia/commons/a/a9/World_map_colored_using_the_four_color_theorem_including_oceans.png1

## THE FOUR COLOR THEOREM

When trying to figure out why such a phenomenon works, it's best to formally state it to avoid confusion or ambiguity. The intuitive definition of what is called the four color theorem is:

> *In mathematics, the four color theorem, or the four color map theorem, states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. Adjacent means that two regions share a common boundary curve segment, not merely a corner where three or more regions meet.*[2]

This theorem answers the questions asked previously by saying only four colors are necessary.

As far as anyone knows, the first time this idea was conjectured was in 1852, when Francis Guthrie was coloring a map of England's counties and noticed he only needed four colors to do so, similar to the world map example earlier. Curious, he asked his brother to look into the problem. He took it to his professor at University College London, Augustus De Morgan. Morgan responded saying he doesn't know why or if the conjecture is true. In The Athenaeum, a magazine, "H.G." (who is assumed to be one of the Guthrie brothers) published the conjecture for anyone who could prove it.

There were many failed attempts at proving the conjecture, some of which took over a decade to disprove. It wasn't until over a century later that the conjecture became a theorem.

In the 1970s at the University of Illinois at Urbana-Champaign, a math professor, Wolfgang Haken, and an assistant professor, Kenneth Appel, used the advantage of having modern computers, which was a luxury at the time, to prove the conjecture. To do this, they considered all possible ways regions could be arranged that meet the conditions of the conjecture and checked each one on the computer. Relative to today's computing power, the ones Appel and Haken used were excruciatingly slow; so slow that the calculations took over 1200 hours, or 50 days!

The results proved the conjecture to be true and were verified by various mathematicians. The media worldwide reported the exciting news of the proof of the over century-old conjecture, which subsequently became the four color theorem that we know today. It is the first major theorem to be proved by computers.

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[2] https://en.wikipedia.org/wiki/Four_color_theorem

A computer proof, especially for those who appreciate the elegance of a formal proof, can be unsatisfactory. How can we be sure the theorem is correct without having a computer tell us it is?

Unfortunately, no one has found a formal proof of the four color theorem. It can be so simply stated yet coming up with a formal proof is too difficult even for the best mathematicians in the world, closely resembling the issue mathematicians have with the Collatz Conjecture (a beauty that deserves its own paper). While enjoying an elegant proof of the theorem is not currently possible, some level of intuition can be gained by closely analyzing a map.

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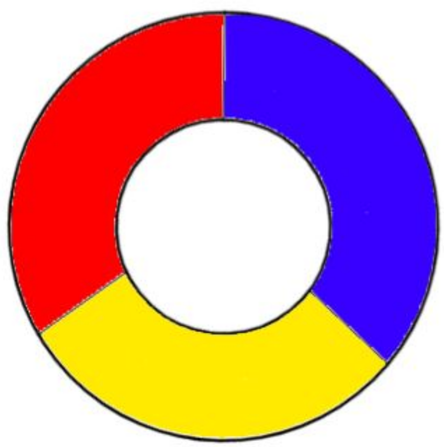
**INTUITION**

The condition of maps described by the four color theorem is that they have contiguous regions with no unclosed curves or suspended lines. If all regions must be colored, there cannot only be one or two other colors since there can be four regions which are adjacent to each other. We will call them interjacent. To make this clear, we can draw a simple map:



The center color cannot be colored red, yellow, or blue since it is adjacent to all three colors. Thus, a fourth color is needed.

While it is true that a map can be drawn such that only two colors (for example, rectangles aligned in a straight line adjacent to each other) or three colors (a circle divided into three regions by three unique radii) are needed, the theorem is concerned with the minimum number of colors needed for *any kind of map*.

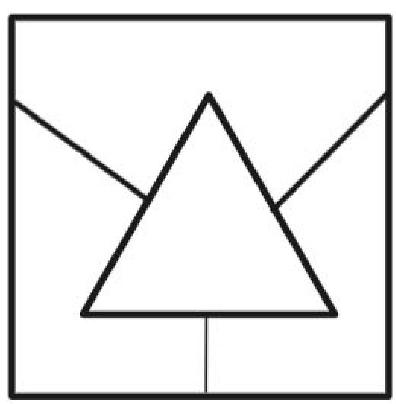
The problem arises when trying to come up with an example where four colors are not sufficient and five colors are absolutely necessary. Such a map requires that there are five regions that are interjacent. Can there be five interjacent regions?

Based on the four color theorem, the answer is no. But how can we give intuition for that (not a proof, as that would require more rigor)?

To show this, we can take an approach that topologists often take in three dimensions: allow for stretching of boundaries. Regions in a map defined by the four color theorem can have boundaries that are straight or curved, so long as the region is an enclosed figure, no new regions are created and no existing ones are destroyed. The original shape and final shape after molding are said to be homeomorphic.

This idea is crucial because we can treat all regions as regular polygons which can later be transformed into any shaped region. This is to simplify the boundaries created and make the explanations clearer. If we can prove five polygons cannot be interjacent, then no five regions, regardless of shape, can be interjacent by the homeomorphic property.

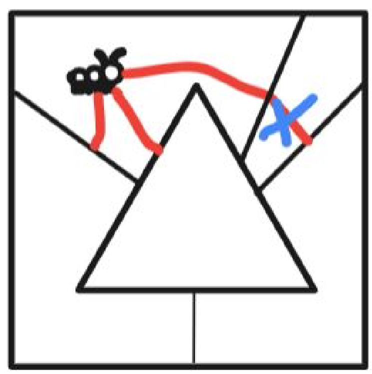
To prove five polygons cannot be interjacent, we can take a rectangle as the map boundary and include a polygon within it such that lines that create boundaries can only intersect the sides of the polygon and not the vertices. We can start with the simplest shape which is a triangle:



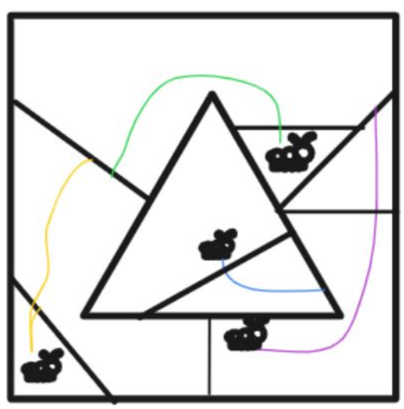
Clearly, there are four interjacent regions and thus four colors are needed. The map consisting of two circles shown earlier is precisely this shape but with molded boundaries (homeomorphic).

In order to create five interjacent regions, a region must be drawn such that it is adjacent to all the regions drawn above. However, this is impossible because drawing a new region requires the division of one of the existing regions into two new regions. Doing so will always create a region that is not adjacent to at least one other region. This can be visualized by imagining an ant walking along this map. He starts off in one of the regions but wants to travel to another region without having to go through a different region.

He can do so because the regions are interjacent. However, by dividing one of the regions, his path will be interrupted by another region.



Now imagine an infinite number of ants located on the map. No matter where you put the new boundary, it will always cut the path of at least one ant so it can't directly access one of the other regions without going through another.



Therefore, five interjacent regions are not possible. Knowing this, there is no situation where five colors are necessary to color in regions such that no two regions are adjacent. If there are five regions in a map, there necessarily are two regions that are not adjacent so they can be colored with the same color.

What does this mean for the four color theorem? This certainly does not prove it, but it shows that at least the subset regions of a map comprising five regions does not need five colors. It does not, however, mean that maps with more than 5 regions will most certainly only need four colors. Yes, it works on a local scale but proving it on a global scale is much more difficult. This is the reason why the four color theorem hasn't been proven without a computer yet. Doing so will probably win the person who finds the proof global recognition and awards.

What practical use does the four color theorem have? Realistically, none that changes the world, but that does not change its significance. It shows the beauty of mathematics and gives a light introduction to topology, which itself has practical applications. It also showed the skeptical mathematical world that computing can impact pure mathematics. Nowadays, it is routine to use computers for proofs.

Perhaps the most direct practical use of the theorem is grabbing a coloring book, four different colored pencils and getting to work. It's a nice way to pass time when you need to.

*All images were created by Rowida Zaid unless otherwise indicated by a footnote.*