Mathematical essay

Carl Friedrich Gauss

A brief view over Abstract algebra and Algebraic number theory

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Évariste Galois

Wahrlich es ist nicht das Wissen, sondern das Lernen, nicht das Besitzen sondern das Erwerben, nicht das Da-Seyn, sondern das Hinkommen, was den grössten Genuss gewährt.

- Carl Friedrich Gauss (1808)

Introduction

Algebraic number theory, as a branch of number theory, uses techniques of abstract algebra to study integers and rational numbers ($\mathbf{Z}$ and $\mathbf{Q}$), and their generalizations. In this essay, I will discuss my mathematical passions, elements of the history of abstract algebra and algebraic number theory; the mathematicians who studied and published works on the topic, and also my personal views for the modern theory itself.

1. My personal study and mathematical interests

I was born in Bulgaria and my mathematical journey began in Primary School, where I was firstly introduced to the natural numbers $\mathbf{N}$ and the elementary operations - addition, subtraction, multiplication and division.

Throughout my early years in school, mathematics was not a significant challenge for me. I started feeling passion for it in the later stages of my student life. My discovery of Calculus at the age of 14 inspired me on the subject and brought my further interest, to a point where I was simply captivated by it. The desire to learn more abstract concepts made me walk down the road of the complex world to the advanced mathematics. Thanks to my knowledge of Calculus, I was able to learn some principle concepts about linear algebra, complex analysis, real analysis, topology, abstract algebra, even though I never studied them in detail, as I am still preparing for my A levels.

Currently, I am finishing my studies on Multivariable Calculus and after that, I will be seeking to further improve my knowledge of mathematics in the field of linear algebra. With that, I am hoping to one day have the necessary knowledge to tackle different challenges in modern abstract algebra.

It is my dream to study mathematics in a prestigious university, where I could do research on the field of pure mathematics, at a high research level.

2. The innovative contributors to abstract algebra

In pursuit of further knowledge, mathematicians throughout the centuries were trying to solve multiple problems and prove many theorems, and by this they introduced new, more elegant and efficient methods for solving the mathematical challenges. They made revolutionary discoveries, which contributed to the progress in the different fields of mathematics, and are being worshiped today for their significant contribution.

An early innovator to the field of analytic number theory is Leonhard Euler, who lived in the 18^{th} century. He used analytic methods to solve problems in number theory, hence uniting the two different branches into one new field of study that is called analytic number theory. His work is later used by other mathematicians, who further developed his methods and ideas.

An innovator to the field of abstract algebra and number theory was Carl Friedrich Gauss. His work on proving the fundamental theorem of algebra, which states that every single-variable polynomial with complex coefficients has at least one complex solution, is elegant - it was his doctoral dissertation. His later works, such as introducing the congruence sign and modular arithmetic, and also proving Fermat's last theorem for integer value of $n=5$, were very important to the field of number theory. Some of those works created methods and techniques, which are used in the modern world of algebraic number theory.

In the early 19^{th} century, the young french mathematician Évariste Galois determines the necessary condition for a polynomial to be solved by radicals. He made significant contributions to two major branches in abstract algebra - group theory and also laid the foundations to what is known today as Galois theory. Galois was also able to prove the impossibility of a "quintic formula" by radicals - in other words, that fifth degree and higher degree polynomials are not solvable by radicals.

There are also several modern mathematicians of the 20^{th} century, whose contribution to the field of abstract algebra I would like to point out. One of them is the German mathematician David Hilbert, whose work on invariant theory and algebraic number theory is of tremendous significance. He and his students also contributed for the establishment of rigor and valuable tools in the modern mathematical physics.

The German mathematicians - Emil Artin and Emmy Noether were one of the leading mathematicians in the early and middle 20^{th} century. Artin is best known for his work on algebraic number theory and he contributed to the class field theory. In addition to that, he made his contribution in the field of pure mathematical theories of rings, groups and fields. Significant is also Emmy Noether's contribution to the field of abstract algebra and theoretical physics. She introduced modern methods in abstract algebra and she founded the theory of central simple algebras.

In the late 19^{th} and early 20^{th} century, German mathematician Richard Dedekind made major contributions to abstract algebra. In particular, ring theory, algebraic number theory and also the definition of the real numbers $\mathbf{R}$. He was one of the first mathematicians who gave lectures on Galois theory, and he also was able to understand the importance of the notion of groups for algebra and arithmetic.

Mathematicians throughout the last few centuries contributed with their work for the development of modern abstract algebra. They gave rigorous proofs on certain conjectures, which were unsolved for many centuries, and gave further extensions to the different mathematical case studies. Of high importance are rings, groups, fields, and the notions in Galois theory.

Many discoveries, even though not directly related to abstract algebra, were made by mathematicians like Bernhard Riemann, Peter Gustav Lejeune Dirichlet and others. They did contribute to number theory however, which is an important field of mathematics in today's modern world.

It is my strong belief that abstract algebra plays a significant role in today's world and thanks to the hard work of the mathematicians of the past generations, we are able to deepen our understanding of it and use the established rules to solve some of the most challenging problems in pure mathematics.

3. My personal views on the modern abstract algebra

I personally have studied about the research and contributions made by many incredible mathematicians of the 20th century, all who played a role to the proving of Fermat's last theorem. It is one of the most challenging theorems to prove, which was only proven in the 1990's by Andrew Wiles.

The theorem, while it was hard to prove, is pretty simple to describe, as the following conditions have to be satisfied:

i) There are no integer values $n > 2$, for $n \in Z^{+}$, such that the equation $x^n + y^n = z^n$ is true,
ii) n is a non-zero integer, otherwise that will be a trivial solution,
iii) x, y and z are also non-zero integers.

In order to prove Fermat's last theorem, Andrew Wiles actually proved the modularity theorem for semistable elliptic curves, which implied that Fermat's theorem was true. This theorem, also known as the Taniyama–Shimura conjecture or Taniyama–Shimura–Weil (thanks to Andrè Weil), was later fully proven by several other mathematicians, who further extended the techniques of Wiles. By doing that, they proved the full modularity theorem

The Taniyama–Shimura–Weil conjecture is a special case of the more general conjectures, due to Robert Langlands. The Langlands program seeks to attach a suitable generalization of a modular form to more general objects of arithmetic algebraic geometry. Most of these extended conjectures have not yet been proven and are challenging for the ones who study the program and would like to work on them.

My view on abstract algebra and algebraic number theory is that they are something modern society requires. The stepping stones, which mathematicians of the past built, allow modern mathematicians to dwell further into the unknown and develop rigorous solutions and proofs to some of the most challenging problems in pure mathematics. The Langlands program - one of the combining generalized programs - is trying to generalize specific objects and provide a translation between different topics in pure mathematics, which are not easily found related. On the other hand, Galois theory, not only plays a significant role for the modern proofs in algebra, but is also used in areas like cryptography, physics and others.

A subject of algebra is the study of the unknown and I strongly believe that understanding certain relationships and finding patterns, will enhance our understanding of the subject even more, allowing for future discoveries to be made, which will be of great importance to the world we live in.