The Bar Problem

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1 Abstract

This essay will focus on an optimization problem: what is the most efficient path to take when queuing for a drink at a bar? Personally, I've found it to be a hard decision. Do you go straight towards the bartender which will be swamped with people? Or do you go round to the edges where the bartender is far away? Is there a compromise between these two choices?

What follows is an a priori generalisation of the problem in terms of formulae based on a postpriori experience. Instead of using the formulae algebraically, Python was used to produce graphs of typical scenarios which were analysed to find the most efficient entry point into the crowd. Typical scenarios are discussed on page 4.

It was found that, depending on certain common scenarios, the best path to get a drink can be non-trivial. In addition, the solution found for a particular scenario is only beneficial to a single person, and is detrimental to everyone involved if the general crowd follows the same recommendation.

2 The Bar Problem

To approach this problem, it will be broken down into two main sections: the approach to the bar counter from an entry point in the crowd, and the wait to be served at the counter. These sections will then be combined to find the best entry point to join the group huddled around the counter. Firstly, the problem will be described and assumptions explained.

2.1 Overview

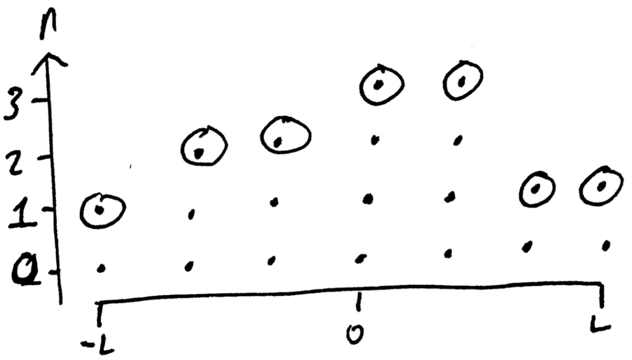

Figure 1: An example scenario of 4 layers of people spanning the length of the counter. The outer edge of the crowd is denoted by $e(x)$; the nodes encircled.

Let $e(x) : \mathbb{N} \to \mathbb{N}$ describe the edge of the grouped people where an entry point can be chosen. Let x be the distance parallel to the centre of the counter. The counter length is $2L$.

Only at $n = 0$ can customers can be served. Once served, the person shall vanish (realistically they would slide through the nodes) creating an empty space. We shall consider the N people being served at the same time to be an iteration; where N is the number of bartenders. Only after an iteration can people progress towards the counter.

Below the discrete points generated by $e(x)$ is a graph where each node represents a queuing customer heading towards the counter. The nodes are placed in a grid pattern. It will be assumed that if there exists an empty node, it will only be taken up by the person directly above that node. Subsequently, the empty node created from the person moving towards the counter is filled. Having this consideration in place limits our problem to only big crowds. The limitation is realistic as people will generally head straight towards the counter in a large crowd.

Now we have constructed a notion of progress, and therefore a way to measure the most efficient entry point := the minimum expected iterations from entering at $e(x)$.

Once at the counter, there will be an expected number of iterations before being served by any bartender. To find this expectation, first the probability of being served at the counter given the position of the bartender(s) must be found. Let $s(x) : \mathbb{N} \to \mathbb{R}$ be the probability of being served at x. Thus, the expected iterations $E(x)$ will be the sum of the expected iterations of following the vertical path from $e(x)$ to $n = 0$ and the expected iterations waiting to be served at the counter. We can then find the best spot to enter the crowd given $e(x)$ and the displacement of the bartender(s) from the origin.

We can now formalise the problem.

2.2 Approaching the counter

Let us define the expected iterations until the counter is reached from $e(x)$.

Our function $s(x)$ gives the probability of one being served at the counter at x. Progress of moving forward can only be made in a column, so if the person at $n = 0$ is served then everybody in the column moves down one node. If one enters at $e(x)$, then it will take $e(x)$ successive trials until the counter is reached (figure 2). As expected outcomes are linear they can be summed until they reach layer $n = 0$. The expected outcome of a geometric probability is given by the inverse of the probability. So the average iterations to get to the counter is given by

$$\underbrace{\frac{1}{s(x)} + \frac{1}{s(x)} + ...}_{e(x)} = \frac{e(x)}{s(x)}$$

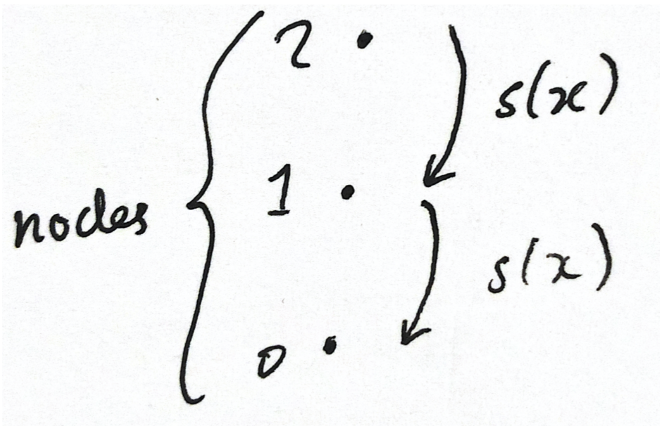

Figure 2: People moving down the nodes $2 \to 1 \to 0$ with a probability of $s(x)$ each.

2.3 Waiting to be served

We shall now define $s(x)$. The approach taken will define a probability distribution function $p(x) : \mathbb{R} \to \mathbb{R}$. Consequently, $s(x)$ will be the sum of all points $p(x)$ that an individual at the bar is taking up at the counter. Suppose that there is a inverse proportionality with a vector distance to the bartender(s), denoted by b_i, and the person at the counter x. These bartenders are placed at $n = -1$ so the vertical distance from the bartender b_i to the customer is 1. So

$$p(x) \propto \sum_{i=1}^{N} \frac{1}{\sqrt{(b_i - x)^2 + 1}} \tag{1}$$

$$= c \sum_{i=1}^{N} \frac{1}{\sqrt{(b_i - x)^2 + 1}} \tag{2}$$

where c is a proportionality constant. We shall need to normalise $p(x)$ so it gives the chances of being served within a distance $2L + 1$ to be 100% (the addition of 1 becomes clear when considering the probability of a discrete unit of space later on):

$$a\int_{-L-\frac{1}{2}}^{L+\frac{1}{2}} p(x)dx = 1 \tag{3}$$

$$\Rightarrow a\int_{-L-\frac{1}{2}}^{L+\frac{1}{2}} c\sum_{i=1}^{N} \frac{1}{\sqrt{(b_i - x)^2 + 1}}dx = 1 \tag{4}$$

$$\Rightarrow ac\sum_{i=1}^{N}\int_{-L-\frac{1}{2}}^{L+\frac{1}{2}} \frac{1}{\sqrt{(b_i - x)^2 + 1}}dx = 1 \tag{5}$$

$$\Rightarrow \quad ac\sum_{i=1}^{N} \left[\sinh^{-1}(b_i - x)\right]_{-L-\frac{1}{2}}^{L+\frac{1}{2}} = 1 \tag{6}$$

We have obtained the normalisation constant

$$a = \left| \left(c\sum_{i=1}^{N} \sinh^{-1}(b_i - L - \frac{1}{2}) + \sinh^{-1}(b_i + L + \frac{1}{2}) \right)^{-1} \right|$$

We take the absolute value of a as the second term will be greater than the first; which would result in a negative probability. Now our normalised probability distribution function $p(x)$ can be used to find the probability of getting served at x. We shall assume that the people spanning the length of the counter take up a unit x of space. So, the probability of being served at a node $d \in \mathbb{N}$ in the row $n = 0$ will be the sum of the probability over the distance $[d - \frac{1}{2}, d + \frac{1}{2}]$. This gives

$$a\int_{d-\frac{1}{2}}^{d+\frac{1}{2}} p(t)dt \tag{7}$$

$$= a\int_{d-\frac{1}{2}}^{d+\frac{1}{2}} c\sum_{i=1}^{N} \frac{1}{\sqrt{(b_i - t)^2 + 1}}dt \tag{8}$$

$$= ac\sum_{i=1}^{N}\int_{d-\frac{1}{2}}^{d+\frac{1}{2}} \frac{1}{\sqrt{(b_i - t)^2 + 1}}dt \tag{9}$$

$$= ac\sum_{i=1}^{N} \left[\sinh^{-1}(b_i - t)\right]_{d-\frac{1}{2}}^{d+\frac{1}{2}} \tag{10}$$

$$\Rightarrow s(x) = ac\sum_{i=1}^{N} \left(\sinh^{-1}(b_i - d - \frac{1}{2}) - \sinh^{-1}(b_i - d + \frac{1}{2}) \right) \tag{11}$$

Note how the proportionality constant is cancelled out by the normalisation. Only the fundamental measurement of distance can change the proportionality.

Waiting to be served at the counter is a geometric probability as described before; you are waiting a number of tries until a successful iteration. Therefore the expected number of iterations until one is served at the counter is given by $\frac{1}{s(x)}$.

2.4 Optimisation

The expected outcome to arrive at the counter and then proceeding to wait gives the average iterations for one person to be served. But for each iteration, N people are being served at a time e.g. if there are two bartenders, the expectancy decreases by a half. Therefore, the total expected iterations for being served will be

$$E(x) = \frac{1}{N}\left(\frac{1}{s(x)} + \frac{e(x)}{s(x)} \right) \tag{12}$$

If we differentiate $E(x)$ and find the minimums then we can find at which point $e(x)$ we should enter to have the least expected iterations:

$$
\begin{aligned}
& & E'(x) = 0 \quad (13)\\
\Rightarrow & & \frac{d}{dx}\left(\frac{1}{N}\left(\frac{1}{s(x)} + \frac{e(x)}{s(x)}\right)\right) = 0 \quad (14)\\
\Rightarrow & \frac{1}{N}\left(-\frac{s'(x)}{s(x)^2} + \frac{e'(x)s(x) - e(x)s'(x)}{s(x)^2}\right) = 0 \quad (15)\\
\Rightarrow & & -s'(x)(1+e(x)) + e'(x)s(x) = 0 \quad (16)\\
\Rightarrow & & s'(x) - \frac{e'(x)}{1+e(x)}s(x) = 0 \quad (17)\\
& & (18)
\end{aligned}
$$

This is a first order differential equation with a solution for $s(x)$ being:

$$
\begin{aligned}
s(x) &= b\exp\left(\int \frac{e'(x)}{(1+e(x))}dx\right) \quad (19)\\
&= b\exp\left(\ln|1+e(x)|\right) \quad (20)\\
&= b(1+e(x)) \text{ where } b \text{ is a constant} \quad (21)
\end{aligned}
$$

We have obtained a compact formula to find stationary points of $E(x)$.

$$s(x) = b + be(x) \quad (22)$$

Thus, minimum expected iterations can be found using eq. 22 and by checking $E(-L)$ and $E(L)$.

The general problem has now been given. The solution is tedious if we were to solve for x algebraically, so we shall use Python to find the minimum expected value(s). Hence, we can investigate specific scenarios that are most likely to occur; what I have experienced as a student myself on a busy night out.

2.5 Discussion

Suppose that there are two bartenders covering a 6 metre counter; $L = 3$ and centred at $x = 0$. For optimum coverage, consider the bartenders to be equidistant, so $b_1 = -1$ and $b_2 = 1$.

We will vary $e(x)$ and create a table where the counter and bartender is constant. Let us explore:

1. A rectangular block of people; $e(x)$ is constant
2. A crowd focused on one side; a linear slope
3. A concentrated huddle around the bartenders; a Gaussian distribution centred at $x = 0$.

The Python program will produce 3 graphs displaying: the probability distribution $p(x)$; the edge of people waiting $e(x)$; and the expected number of iterations until N are served from joining the group $E(X)$.

The graphs can be found starting at page 6.

We can analyse these graphs with a table:

Figure	$e(x)$	Best Entry points at x	Min Iteration	Max Iteration	Range of Iterations	Size of Range
3	4	$-1, 0, 1$	14	28	$[14, 28]$	14
4	$-x + 4$	1	11	45	$[11, 45]$	34
5	$\lfloor 4e^{-(0.16x^2)} \rfloor$	$-3, 3$	11	14	$[11, 14]$	3

By analysing this table, we can make several interesting points given the initial conditions.

Plausibility of Expected Outcomes. Except from using our intuition when looking at the graphs, we can quickly do some rough calculations to confirm the plausibility of the expected iterations calculated. Take figure 3 as the trivial example. The number of people is $4 \times 7 = 28$. Comparing to the average wait time, being bounded between 14 and 28, one is expected to be served within the first round of people which is a reasonable estimate.

This method does not agree with figure 4 nor figure 5, but they have a non-trivial crowd formation.

Best Point to Enter. Limited data from limited scenarios has been presented, but from these graphs it is best to head straight to the nearest available bartender in general. Although, if the bartender is swamped with people, it is best to enter from the side.

Optimisation for the Bar. By observing the range of only these three typical scenarios, the iterations can vary massively thus the time to wait can vary massively. Although, it can be argued that if the crowd huddles around the bartenders (figure 5), it will actually benefit everybody in general. With swamping the bar, there are fewer people to be served in general and there is a greater density of people around the bartender(s). Therefore, the bulk can be served quickly - decreasing the average iterations.

Most importantly, even though it is suggested to go the edges of the bar, if everybody did this we would result in figure 3. This would increase the range and minimum iterations for everybody; resulting in a quadrupled average waiting time.

Futher Work. The derived equation had to be graphed for these scenarios. To solve equ. 22 directly would require further work, but could possibly give interesting results for optimisation. The complexity of the equations could be reduced by cleverly choosing the proportionality between distance of bartender and the chances of getting served. Instead of a vector distance ($\sqrt{x^2+1}$), maybe simply squaring the x-components (x^2+1) would suffice, or just taking the difference, or even using a Gaussian distribution. Several of these were attempted, but they resulted in complicated functions to handle that would require a more thorough analysis than presented in this essay. However, it was noted that these functions could be expanded into their respective MacLaurin series, giving a rough approximation in terms of polynomials. Polynomials are relatively easy to differentiate, although it would lead to multiple roots when solving for equ. 22 which could be challenging.

3 Conclusion

For further work, I would focus on the Gaussian distribution of people $e(x)$ (figure 5) and work with equ.22. Variables such as the number of bartenders, displacement of bartenders and counter length could be altered. More variables could also be introduced.

From experience, the most typical crowd is one that huddles around the bartender where the most efficient route is approaching the counter from the sides. If there is no obvious crowd, heading straight towards the bartender results in the quickest drink instead. Using my familiarity of bars as empirical evidence, this essay has successfully produced a rough mathematical model able to give answers for the most efficient path(s) to take when queuing for a drink.

Moreover, this essay has shown how the same recommendation can reduce the waiting time for a single person but ultimately lead to an increased waiting time for the majority. This was a result that I had not previously considered before investigating this problem.

I hope the reader has been enlightened to a new concept, re-thinking how they queue at the next bar they visit - hopefully resulting in a faster drink.

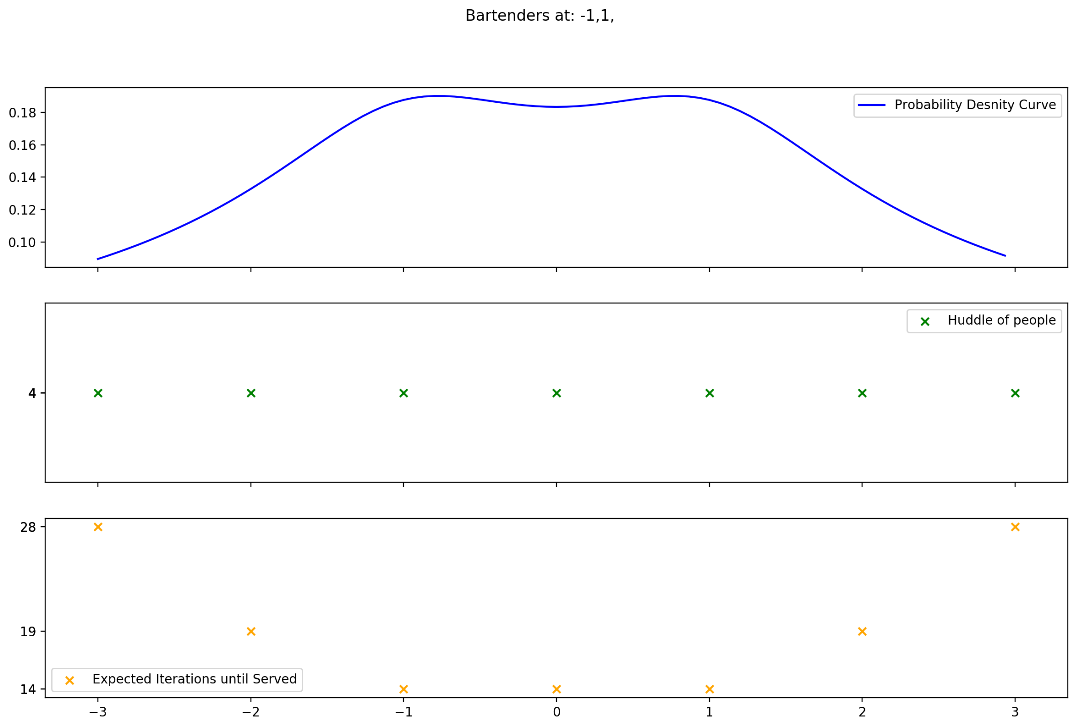

Figure 3: A trivial example which shows that $E(x)$ is working as expected.

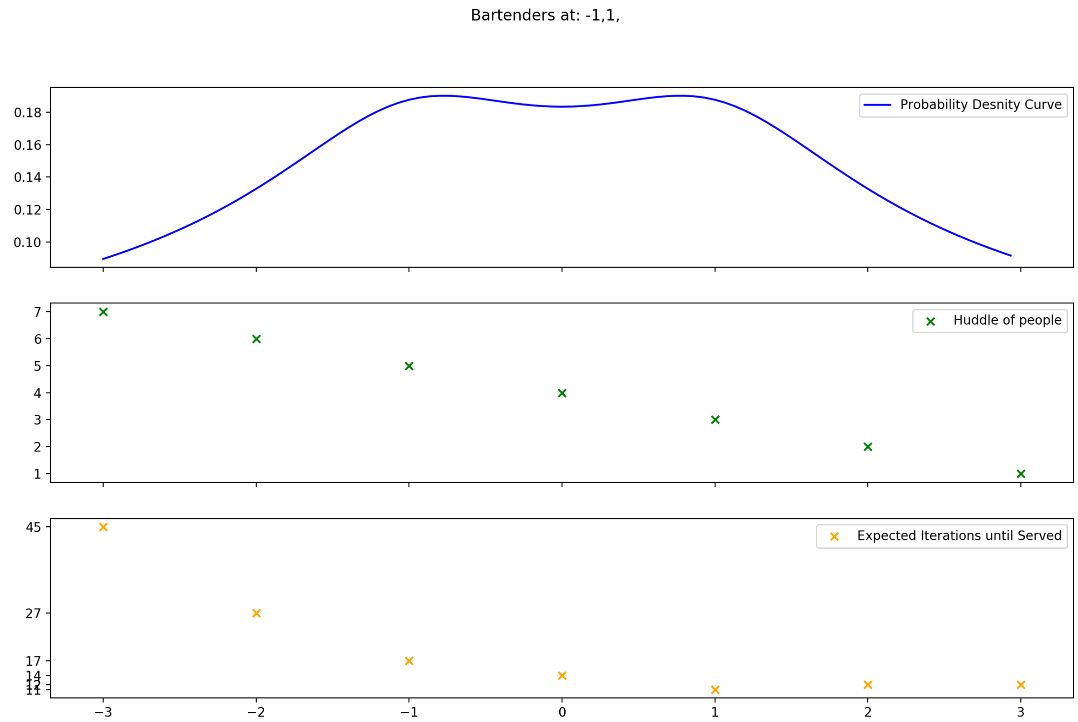

Figure 4: A not so trivial example. Heading towards the bartender is the best option which again, confirms the plausibility of the formula.

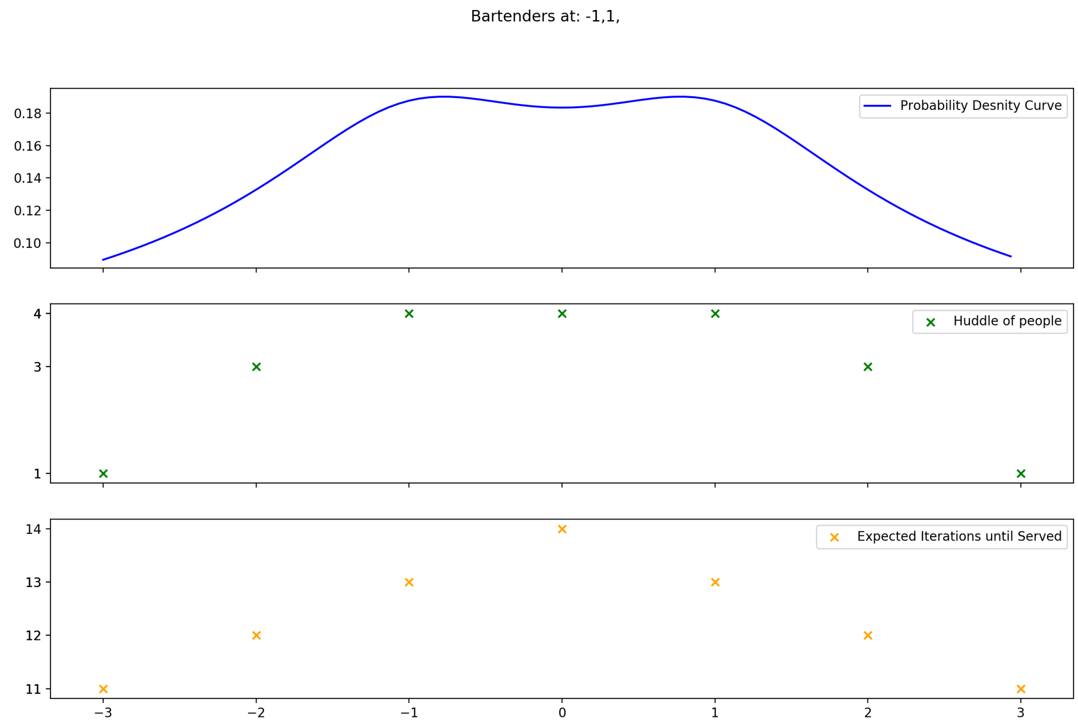

Figure 5: A non-trivial example, where the best option is to head furthest away from the bartenders.