We have all been there. In the playground, fighting with a friend. However, this was no physical conflict. It was far more serious. Who could name the largest number? Early battles would have been won with numbers such as 100. As the years went by, numbers in the millions, billions and possibly even trillions would bring victory. However, a time would come when someone would discover a weapon which would be unbeatable in this number war. The huge, indescribable, colossal number that is... infinity. However, our young brains were underprepared to tackle this vast, complex and extensive topic of mathematics. Our statement that infinity is a number, and the automatic, retaliatory response of, “what about infinity plus 1?”, were undoubtedly wrong mathematically. In fact, infinity is not a number. It is an idea. It is a concept of something that has no end. For example, here are some really big (and I mean really big) numbers that are known. A googol is 10^100- that is 1 followed by 100 zeroes. A googol, which is larger than the number of atoms in the known universe, is dwarfed by a googolplex (10^10^100)- which is 1 followed by a googol zeroes. This number is so big that there is not enough space in the known universe to write out a googolplex. But we can go bigger still. Much bigger. Graham’s Number, for example, is tremendously, incomprehensibly, immensely larger than a googolplex. However, all of these numbers are finite. Eventually, they terminate. So, they are not even close to infinity. Why? Because infinity is endless.

This is hard to get our heads around because there is nothing in the observable universe which is endless. Several mathematicians throughout history have simply refused to accept the existence of infinity. As the ancient Greeks developed the studies of mathematics and science, they very soon came across the concept of infinity. Can matter always be divided into smaller and smaller pieces indefinitely? Does the universe have an end? How many natural numbers (1,2,3,4,...) are there? The Greeks, in general, feared the idea of infinity and tried to avoid it. Instead the idea of a ‘potential’ infinite was offered to solve some of these problems. Introduced by Aristotle, the idea is that we can never conceive of the natural numbers as a whole, but given any natural number, another can be found, so they are potentially infinite. So, Euclid did not actually prove that the prime numbers are infinite in 300BC. Euclid’s famous theorem actually stated that, ‘Prime numbers are more than any assigned magnitude of prime numbers’. The infinite was carefully avoided. Since then, for hundreds of years, little progress was made on the actual infinite. However, the careful treatment of infinite processes by Greek mathematicians, laid the foundations for for the rigorous treatment of infinite processes in the development of calculus in the 19th century. For example, the method of exhaustion was developed formally by Greek mathematician Eudoxus. This technique was used to rigorously prove the area of a shape by inscribing inside it a sequence of polygons with an increasing number of sides, whose areas converge to the area of the containing shape. These ideas were seen as a precursor to the methods of calculus. While differential calculus is used to study the rate of change of a function, integral calculus is used to find the area under the curve of a function. Integral (pardon the pun), to the study of calculus is infinitesimals. Infinitesimals are things so small that there is no way to measure them. The insight with exploiting infinitesimals was that they can still retain certain specific properties, such as angle or slope, even though they are extremely small. Infinitesimals are infinitely small.

Zeno of Elea, the Greek philosopher, was the one of the first to really think about the implications of infinite processes and the infinitely small. He put forward Zeno’s Dichotomy paradox. Suppose you want to cross a room. In order to get to the other side, you must first get to the halfway point, which will take you some finite amount of time. Before you can get to the other side, you have to cross half of *that* distance, at which point you would be a three-quarters of the way across. Before that, you would have to cross half of that quarter, and so on, infinitely. Each of these steps must take a finite amount of time. And yet, you have to cross an infinite number of distances to walk across the room—or indeed any distance at all. Since one cannot travel an infinite number of distances in a finite period of time, motion itself is impossible. Of course, motion is possible. It does not take each finger an infinite amount of time to reach the keyboard to type each letter of this essay. The solution lies in the fact that infinite series can sum to a finite answer. To solve Zeno's Dichotomy paradox, an infinite series can describe your journey as you walk across the hall. Assuming you are walking at a constant speed, each 'part' of the journey takes half as the long as the previous 'part'. If the first part of the journey takes one second, then the infinite series is:

$S=1+1/2+1/4+1/8+...$

$S= 1+1/2+1/4+...$
$0.5S= 1/2+1/4+...$
$S-0.5S=1$
$0.5S=1$
$S=2$

So, it would take you 2 seconds to walk across the hall.

This seems sensible. However, infinite series that do not converge (each term is smaller than the previous term) in the usual sense can be summed to produce answers that seem to defy basic logic. For example, it would be sensible to say that the sum of all the natural numbers is positive. It would also seem sensible to say that whatever the sum is, it must be a whole number. Another sensible claim would be that it is extremely big and tends to infinity. However, every single one of the previous statements are false. The sum of the natural numbers: $1+2+3+4+... =-1/12$. Edward Frenkel, a mathematics professor at the University of California stated that,
"This calculation is one of the best-kept secrets in math,"
In the following proof, the infinite set A (Grandi's series)= $1–1+1–1+1–1\cdots$; the infinite set B= $1–2+3–4+5–6\cdots$; and the infinite set C= $1+2+3+4+5+6\cdots$(the set of all natural numbers)

$A = 1–1+1–1+1–1\cdots$
$1-A=1-(1–1+1–1+1–1\cdots)$
$1-A =A$
$1 = 2A$
$1/2 = A$

$A-B = (1–1+1–1+1–1\cdots) — (1–2+3–4+5–6\cdots)$
$A-B = (1–1+1–1+1–1\cdots) — 1+2–3+4–5+6\cdots$

$A-B = (1–1) + (–1+2) +(1–3) + (–1+4) + (1–5) + (–1+6)\cdots$
$A-B=0+1–2+3–4+5\cdots$

$A-B = B$
$A = 2B$
$1/2 = 2B$
$1/4 = B$

$B-C = (1-2+3-4+5-6\cdots)-(1+2+3+4+5+6\cdots)$

$B-C = (1-1) + (-2-2) + (3-3) + (-4-4) + (5-5) + (-6-6) \cdots$

$B-C = 0-4+0-8+0-12\cdots$

$B-C = -4-8-12\cdots$

$B-C = -4(1+2+3)\cdots$

$B-C= -4C$

$B = -3C$

$1/4 = -3C$

$1/-12 = C$ or $C = -1/12$

Therefore, $1+2+3+4+...=-1/12$ (incredible!)

In these equations the concept of countable infinity is used. This is a type of infinity that deals with an infinite set of numbers, but if given enough time you could count to any number in the set. It allows the use of some of the regular properties of sums like commutativity in the equations.

The concept of a countable infinity was introduced by Georg Cantor, when he created set theory and in doing so revolutionised the mathematical study of infinity. In set theory, the cardinality of a set of numbers is the number of elements the set has. For example, the cardinality of the set: {1,2,3,4,5} is 5. Cantor said that the cardinality of the set the natural numbers is infinite. However, the real importance of Cantor's work was that not all infinite sets were the same size. There are different sizes of infinity. A set is said to have the same cardinality as another, if each element in one set can be paired to an element in the other set. This was observed by Galileo Galilei around 300 years before Cantor, when he found that the natural numbers can be put in one to one correspondence with the square numbers. So the sets of the natural numbers and square numbers have the same cardinality, which I find to be pretty weird and counterintuitive. However, it can be shown geometrically. Consider two concentric circles, one with a greater circumference than the other. Each has infinitely many points on its circumference, yet the outer circle, since it is bigger, seems to contain more points. Now consider a radius which sweeps along the two circles. Each time it passes through a point on the larger circle it also passes through a point on the smaller circle. So, the two circles contain the same number of points, even though one is larger than the other.

Cantor took Galileo's ideas one step further, showing that the set of natural numbers, which is infinite, cannot be put into one to one correspondence with the real numbers (the numbers that form the number line), which is also infinite. Introduced earlier, the natural numbers are said to be countably infinite- given enough time, hypothetically any number in the set can can counted to. However, the set of of real numbers is said to be uncountably infinite. Assume that we have a list of all the real numbers from 0 to 1 written down one on top of the other. Create a number by taking the number in first decimal place from the first number in the list. This will be the number in the first decimal place of your number. Then take the number in the second decimal place of the second number in the list, which will be the number in the second decimal place of your number and so on. Now take your number and add one to each digit to form a new number (9 goes to 0). Therefore, we have created a number that is not in our 'complete' list, as it differs from the first number, in the first digit, the second number in the second digit and so on. So, our initial assumption that we have a complete list of all of the real numbers between 0 and 1 has arrived at a contradiction, so the set of real numbers is uncountably infinite.

Cantor's work on infinite sets led to him hypothesising that there is no set whose cardinality is strictly between that of the integers and the real numbers. However, Cantor could not prove it and the so called Continuum Hypothesis remains unsolved to this day. Arguably one of the most profound unsolved problems in mathematics, it has led mathematicians to question the very axioms on which their subject is based. Proofs by Kurt Gödel and Paul Cohen found that the continuum hypothesis could neither be proved, nor disproved using the Zermelo-Fraenkel axiom system - which is widely regarded as the most common foundation of mathematics. So, mathematicians have arrived at a crossroads. Should a new addition be introduced to the axiomatic system which can solve this problem? There are two main contenders: forcing axioms and the inner model axiom. However, set-theorists have no idea on which one should be chosen.

So, it seems that the puzzle of the infinite is providing today's mathematicians with just as much trouble as their Greek predecessors some 2,500 years ago and will continue to bother the community for all of time... endlessly. Possibly to infinity and beyond ?