Roller Coasters and the Usage of Space

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March
2020

When I was just a little boy of sixteen years of age, a deep love for theme parks and roller coasters was ignited and has been burning brightly ever since. At first, I feared the roaring monsters made of steel or even wood in some old-fashioned examples. I feared their speed, their loops, and their twists. Until one Monday, I went to the nearest theme park with a group of friends. But not to worry! School was cancelled that day. That particular park was well known to me, for I had visited it with my family several times prior. Back in the day, near the border of the park, there was a fairly simple ride. Rather boring, actually. It rode in a somewhat simple trajectory; up, down, done. In the vicinity of that ride, there was not much park going on. But the owners had expended the park in the meanwhile quite a bit. Practically next to said uneventful ride, there was now a new roller coaster. Maybe even a "hypercoaster", if you will. It was great! Astonishing acceleration, loops, helical passages, a speed hill, and all that jazz. On that coaster, I fell in love with the whole concept of "'great eights", as the French call them "les grands huits".
There had still been some space between the boring "up-and-downer" and the exciting hyper coaster. What better way to make money off of that extra space but by building another ride? An exciting one with a storyline to make the queue less monotonous and the overall park-experience keep its momentum. But there might have been one small problem: How was the space to be used properly? A smaller ride that fits neatly between the two existing ones or a bigger ride that would invade the space already occupied?
Since bigger seems to be better in the business of entertainment (to which I count theme parks), the latter option was apparently the winner. It thus came to pass that the third ride was built right in between - and yet, not quite in between. And what a success it was! But what about the invasion of the space? Not a problem at all in this case! You see, the thing with roller coasters is the following: they can be linked nicely, meaning that the rails of one coaster can go over and under another's rails. Hence, the park ended up with three roller coasters that are linked, and space was efficiently used. Besides, this configuration may make a rider belief for a split second that their head might suffer from crashing against the rails of a neighbouring coaster. Naturally, this would remain an illusion, although an effective one.

Looking at what was just described after putting on a mathematician's glasses, data and measurements like acceleration, height, the number of visitors per day, etc. may be of interest. What fascinated me are the information associated to this "roller coaster link" that remain after treating each of the three rides rather flexibly. What do I mean by this? For argument's sake, let us simplify the coasters to one-dimensional strings in three-dimensional space. They may even float in mid-air, we don't care for the skeletons keeping them grounded. Now, imagine an entity that could easily manipulate the coasters *without* ripping apart their rails. How could such an entity affect the link? The uneventful first ride might gain some exciting passages in the form of an extreme downhill ride, even a corkscrew! Or they could be stretched and take up more space. Or less. So which are the properties that remain the same?
This question leads us to the notion of an "invariant" - an attribute that is preserved under certain actions.

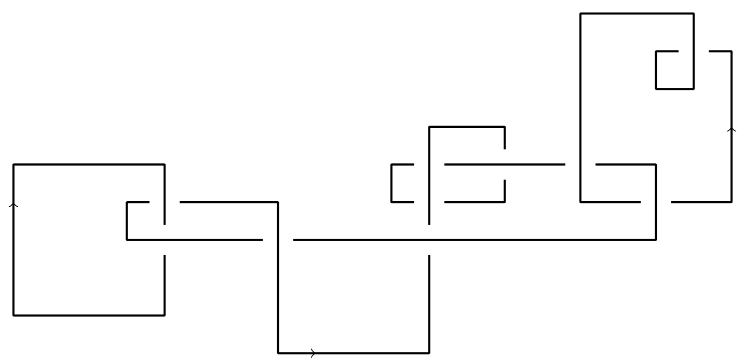

Figure 1: A piecewise linear depiction of the "roller coaster link". Tiny arrows indicate the orientation of each ride.

Given such an invariant of a link as above, one should be able to tell whether two links cannot be transformed into one another by our enigmatic entity: if the invariants associated to two links are not the same, the entity won't be able to manipulate one in order to obtain the other - they are not equivalent. The problem is that given equal invariants to two links does not guarantee that the links are equivalent.
An example of such an invariant would be the *Jones polynomial*. To calculate it for a link diagram L as the one above, one first has to ignore the orientations (the unoriented link shall be called D) and look at the crossings individually. From the unoriented version of the diagram, the *bracket polynomial* $\langle D\rangle$ can be calculated. Each crossing yields two new diagrams with one crossing less each. The bracket polynomials of the two new diagrams D_1 and D_2 are multiplied by an indeterminate A or its inverse A^{-1}, depending on the diagram:

$$\langle D\rangle = A\langle D_1\rangle + A^{-1}\langle D_2\rangle.$$

All in all, the bracket polynomial of a diagram D with n crossings can be

calculated from the polynomials of 2^n diagrams.
In the above example with $n = 10$ crossings, this would result in a total of $2^n = 1024$ polynomials, which is far too many. Consequently, we have to stray from the roller coaster link for an easier illustration. Consider the following link diagram with $n = 3$ crossings. It will be given an orientation (tiny arrows). The oriented diagram shall be called L. Denoted by D will be the same diagram, only without the arrow.

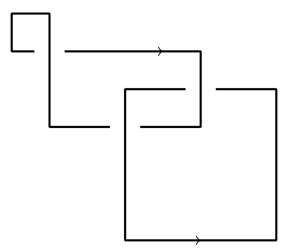

Figure 2: Oriented link L, unoriented link D.

Generating two new diagrams D_1 and D_2 starting from one crossing would mean that an ordering has to be chosen on the crossings. As it turns out, the order of the crossings does not matter! Therefore, we can choose the upper left crossing to be the first one we look at. This crossing can be "erased" in two manners, resulting in the following diagrams:

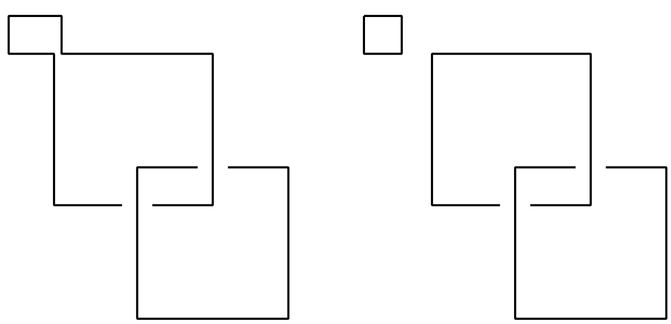

Figure 3: Left: D_1. Right: D_2.

For D_2, a special case occurs: here, we have three components in our diagram, where one of the components is not twisted and disjoint from the other components. The linked components are easily seen to be equivalent to D_1. In a scenario like this, the bracket polynomial behaves as follows:

$$\langle D_2 \rangle = (-A^{-2} - A^2)\langle D_1 \rangle$$

where D_1 is just D_2 without the untwisted component as remarked above.

Continuing this process yields the following bracket polynomial for D:

$$\langle D \rangle = A^{-5} + A^{-1}$$

Note that the bracket polynomial of the *unknot* $\bigcirc$ is $\langle \bigcirc \rangle = 1$. Note as well that the diagram of $\bigcirc$ would be equivalent to a rectangular diagram like $\square$, which is more fitting considering the piecewise linear depictions above.

Unfortunately, the bracket polynomial is not yet what we want an invariant of links to be. There is a type of move that our entity is allowed to perform but which may alter the polynomial. The Jones polynomial of the oriented link L solves this problem.

Going back to the theme park, imagine taking a ride on one of the three coasters, preferably a more exciting one. The direction of the train is the orientation of the ride you are hopefully enjoying. Should you look down and see a train (from any of the three coasters) crossing under your current position from the right, this crossing will be assigned a $+1$. Should you see another vehicle cross under your train from the left, the crossing will be a -1. This can be done for all the crossings of the roller coaster link and the sum is the *writhe* $w(D)$ of the unoriented diagram. In the case of figure 2, we have $w(D) = -3$. The writhe itself is almost an invariant of the link, just like the bracket polynomial is *almost* what we needed. But put them together in a special way and *voilà*! - an invariant pops out.

Given an oriented link L and its unoriented version D, the expression

$$(-A)^{-3w(D)} \langle D \rangle$$

is an invariant of L. Substituting $A^{-2} \mapsto t^{\frac{1}{2}}$ gives us what was implicitly promised earlier: the Jones polynomial

$$V(L) = ((-A)^{-3w(D)} \langle D \rangle)_{t^{\frac{1}{2}} = A^{-2}}$$

- a link invariant as we had mentioned quite often.

All this work can be put together to deliver the Jones polynomial of the example from figure 2 discussed throughout. It is given by $V(L) = -t^2 - t$. A simple polynomial for a simple link. Has it been a wild ride? That is for the reader to tell.

Formulae taken from: W. B. Raymond Lickorish. *An Introduction to Knot Theory*, Springer-Verlag (1997)