

Introduction: The results provided in this paper are considered as homework. If the reader would like the assistance of the author of this paper in answering some arbitrary research question, then please email me at squarecubegalois@gmail.com. Furthermore, you may email me if you have some creative new idea or something. I suppose that I study all areas of mathematics so I don't care about the topic of what you send in. I will email you back if I feel like it, so most emails are probably going to be ignored.

Definition 1: "Let $\varphi(n)$ denote Euler's totient function, where $\varphi(n)$ is defined to be the number of positive integers $\leq n$ that are relatively prime to n , and 1 is relatively prime to all numbers except itself."

Definition 2: "Let $\varphi(n)$ denote Euler's Totient function, then by (Schettler, Jordan)

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product runs over all primes p dividing n ."

Definition 3: "A Carmichael number is an odd composite number n that satisfies the congruence relation of Fermat's little theorem:

$$b^{n-1} - 1 \equiv 0 \pmod{n}.$$

Where b is any integer such that $\gcd(b, n) = 1$ and $1 < b < n$."

Theorem 1: "There doesn't exist any composite number n such that $\varphi(n)|(n-1)$. Equivalently, there doesn't exist any composite number n such that

$$(n-1)/\varphi(n) = a, \quad a \in \mathbb{Z}^+.$$

Where $\varphi(n)$ is defined to be Euler's Totient function."

By (Schettler, Jordan) we know that all possible composite numbers n that can have the property $\varphi(n)|(n-1)$ must be an odd Carmichael number. Thus we will prove the statement of Theorem 1 by mathematical induction on the number of distinct prime factors of n , denoted as c .

For the basis cases, we have $P(c=1)$ and $P(c=2)$. We need to show that there exists no composite number n such that $(n-1)/\varphi(n) = a$, $a \in \mathbb{Z}^+$, where n has one distinct prime divisor for $P(c=1)$ and two distinct prime divisors for $P(c=2)$. However, (Schettler, Jordan) has shown that the only possible composite numbers n that can satisfy the statement $(n-1)/\varphi(n) = a$, $a \in \mathbb{Z}^+$ are Carmichael numbers. And all Carmichael numbers have at least three distinct prime divisors. Thus there exists no composite number n such that $(n-1)/\varphi(n) = a$, $a \in \mathbb{Z}^+$ for the basis cases $P(c=1)$ and $P(c=2)$ since those numbers cannot be Carmichael. Therefore the basis cases of $P(c=1)$ and $P(c=2)$ are true.

For the inductive case assume there exists a positive integer k such that the $P(c = k)$ case is true. In other words, there exists no odd composite Carmichael number n that satisfies the statement $(n - 1)/\varphi(n) = a$, $a \in Z^+$ for $P(c = k)$. Then, for the sake of contradiction for the $P(c = k + 1)$ case, we will assume the contrary to Theorem 1 which gives us

$$(n - 1)/\varphi(n) = a. \quad (no.1)$$

Also, by Definition 2 we have

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad (no.2)$$

$$\Rightarrow \varphi(n) = n \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_k}\right) \left(1 - \frac{1}{p_{k+1}}\right),$$

(no.3)

$$\Rightarrow p_1 \dots p_k p_{k+1} \varphi(n) = n(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1),$$

(no.4)

$$\Rightarrow \varphi(n) / [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] = n / [p_1 \dots p_k p_{k+1}]. \quad (no.5)$$

By construction, we know that $n / [p_1 \dots p_k p_{k+1}] = x$, $x \in Z^+$, since those are all the distinct prime divisors of n . We can then manipulate the line (no.5) to get

$$\varphi(n) / [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] = x. \quad (no.6)$$

Thus we have $[(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] | \varphi(n)$. Looking back at the line (no.4) we get

$$p_1 \dots p_k p_{k+1} / [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] = n / \varphi(n), \quad (no.7)$$

$$\Rightarrow p_1 \dots p_k p_{k+1} / [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] - 1 / \varphi(n) = [n - 1] / \varphi(n). \quad (no.8)$$

By the line (no.1) we can manipulate (no.8) to get

$$p_1 \dots p_k p_{k+1} / [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] - 1 / \varphi(n) = a, \quad (no.9)$$

$$\Rightarrow p_1 \dots p_k p_{k+1} - [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] / \varphi(n) = [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] a. \quad (no.10)$$

By (no.10) we know that $\varphi(n) | [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)]$ since the right-hand side of the equation is an integer and the integers are closed under subtraction. However, we also know that $[(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)] | \varphi(n)$. Thus we have that $\varphi(n) = [(p_1 - 1) \dots (p_k - 1)(p_{k+1} - 1)]$. We can then manipulate the line (no.4) to get

$$p_1 \dots p_k p_{k+1} \varphi(n) = n \varphi(n), \quad (no.11)$$

$$\Rightarrow p_1 \dots p_k p_{k+1} = n. \quad (no.12)$$

Remember that we only resulted in the line (no.12) because we assumed the $P(c = k + 1)$ case is false from the line (no.1). For the $P(c = k)$ case if we were to apply the same arguments to an arbitrary integer y with a number of k distinct divisors then we would get $y \neq p_1 \dots p_k$, since it can only be equal if $(y - 1)/\varphi(y) = a$, $a \in Z^+$, which we have already assumed to not be true by the inductive hypothesis. We can then manipulate the line (no.12) to get

$$p_1 \dots p_k p_{k+1} = n. \quad (no.13)$$

$$\Rightarrow y p_{k+1} \neq p_1 \dots p_k p_{k+1} = n.$$

(no.14)

The line (no.14) then results in a contradiction since y is an arbitrary integer. Thus we have the $P(c = k + 1)$ case holds by contradiction. Therefore, Theorem 1 holds by mathematical induction, contradiction, and properties of Carmichael numbers.

□

Definition 4: “Hofstadter’s Q -Sequence is a recursive sequence that is defined by the following recurrence equation

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2)),$$

where $Q(1) = Q(2) = 1$.”

Lemma 1: “Hofstadter’s Q -Sequence is well-defined for all integers $n \geq 1$.”

First, for notational purposes, we will define the input I of any function to be $I[f(n)] := n$. We will rewrite the sequence as

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2)), \quad (\text{no.15})$$

$$= X + Y . \quad (\text{no.16})$$

Where $X = Q(n - Q(n - 1))$ and $Y = Q(n - Q(n - 2))$. By construction of recursive sequences, we know that if the sequence is defined at some n_0 , then it is defined for all $n < n_0$. This implies that for Hofstadter’s Q -Sequence to be well-defined the input of the terms X and Y must be greater than zero.

$$I[X] = n - Q(n - 1) > 0, \quad (\text{no.19})$$

$$\Rightarrow n > Q(n - 1).$$

(no.20)

$$I[Y] = n - Q(n - 2) > 0, \quad (\text{no.21})$$

$$\Rightarrow n > Q(n - 2).$$

(no.22)

Thus by (no.22) we get that the sequence is well-defined at n iff $n > Q(n - 2)$. Thus we arrive at the following equivalent statement to Lemma 1 “ $n > Q(n - 2)$ for all $n \geq 5$.” Note that we have $n \geq 5$ because $Q(n - 2)$ is not defined for any smaller n . We trivially know that Hofstadter’s Q -Sequence is well-defined for $n = 1, 2, 3$ and 4 , so we do not have to worry about those terms in the sequence. If we rewrite the sequence with a shifted index in n of 1 we get

$$Q(n - 1) = Q(n - 1 - Q(n - 2)) + Q(n - 1 - Q(n - 3)) \quad (\text{no.23})$$

$$= W + Q(n - 1 - Q(n - 3)) \quad (\text{no.24})$$

$$\Rightarrow I[W] = n - 1 - Q(n - 2) > 0 \quad (\text{no.25})$$

$$\Rightarrow n - 1 > Q(n - 2)$$

(no.26)

$$\Rightarrow n > n - 1 > Q(n - 2). \quad (\text{no.27})$$

By mathematical induction we will now prove the statement “ $n > Q(n - 2)$ for all $n \geq 5$.”

For the basis case $P(n = 5)$ we have

$$5 > Q(5 - 2) = Q(3) \quad (\text{no.28})$$

$$= Q(3 - Q(3 - 1)) + Q(3 - Q(3 - 2)) \quad (\text{no.29})$$

$$= Q(3 - Q(2)) + Q(3 - Q(1)) \quad (\text{no.30})$$

$$= Q(3 - 1) + Q(3 - 1) \quad (\text{no.31})$$

$$= Q(2) + Q(2) \quad (\text{no.32})$$

$$= 1 + 1 \quad (\text{no.33})$$

$$= 2. \quad (\text{no.34})$$

By (no.34) we know that the basis case $P(n = 5)$ is true. For the inductive case, assume there exists a positive integer k such that $P(n = k)$ is true. We thus have to show that

$$k + 1 > Q((k + 1) - 2) = Q(k - 1) \quad (\text{no.35})$$

$$\Rightarrow k + 1 > Q(k - 1).$$

(no.36)

By the line (no.27) we have

$$k + 1 > k > Q(k - 1) \quad (\text{no.37})$$

$$\Rightarrow k > Q(k - 1).$$

(no.38)

By the inductive hypothesis that $P(n = k)$ is true, we know that (no.38) is true by (no.27).

Thus (no.38) implies that $k + 1 > k > Q(k - 1) \Rightarrow k + 1 > Q(k - 1)$ is true. Therefore, since the basis case $P(n = 5)$ holds and the $P(n = k)$ case implies that the $P(n = k + 1)$ case holds,

Lemma 1 is true by mathematical induction. □

Definition 5: “A highly composite number is a number of the form

$$N = 2^{a_2} 3^{a_3} \dots p^{a_p}.$$

Where the primes $2, 3, \dots, p$ form a string of consecutive primes, the exponents follow

$a_2 \geq a_3 \geq \dots \geq a_p$, and the final exponent $a_p = 1$, except for the cases where $N = 4$ and $N = 36$ where the exponent is equal to $a_p = 2$. Also, highly composite numbers are numbers such that the divisor function $\sigma(N)$ is strictly greater than for any smaller $n < N$.”

Lemma 2: “Let $\varphi(n)$ denote Euler’s totient function, then we have $\liminf_{n \rightarrow \infty} [\varphi(n)/n] = 0$, where the limit infimum is defined by highly composite numbers.”

Since $\varphi(p) = p - 1$, p a prime, maximizes the totient function, to solve for the limit infimum we must let n be the converse of a prime number with respect to the totient function $\varphi(n)$, which is a highly composite number N . Thus by Definition 2 and letting $n = N$ we have

$$\lim_{n \rightarrow \infty} \inf [\varphi(n)/n] = \lim_{N \rightarrow \infty} \inf [\varphi(N)/N] \quad (\text{no.39})$$

$$= \lim_{N \rightarrow \infty} \inf \left[\frac{N \prod_{p|N} (1 - \frac{1}{p})}{N} \right] \quad (\text{no.40})$$

$$= \lim_{N \rightarrow \infty} \inf \left[\prod_{p|N} (1 - \frac{1}{p}) \right]. \quad (\text{no.41})$$

Since we want to find the smallest possible subsequence then we must minimize $\varphi(n)$ as much as possible. By Definition 5, the prime factors of a highly composite number forms a string of primes $2, 3, \dots, p$, so to minimize $\varphi(n)$ we must let the product operator from the line (no.41) run over all the primes. In other words, we need the product operator from the line (no.41) to run over all the primes because $(1 - \frac{1}{p}) < 1$. This gives us

$$\lim_{N \rightarrow \infty} \inf \left[\prod_{p|N} (1 - \frac{1}{p}) \right] = \lim_{N \rightarrow \infty} \inf \left[\prod_p (1 - \frac{1}{p}) \right] \quad (\text{no.42})$$

$$= 1/\zeta(1) \quad (\text{no.43})$$

$$= 0, \quad (\text{no.44})$$

where ζ denotes the Riemann Zeta function. We can thus recognize that the line (no.44) implies that $\lim_{n \rightarrow \infty} \inf [\varphi(n)/n] = 0$. Therefore Lemma 2 holds by limit laws and Definitions 2 and 5. □

Definition 6: "Let $S_n := \sum_{k=1}^n a_1 r^{k-1}$ be a geometric series with constant ratio r such that $|r| < 1$. Then the partial sum of the series at n is denoted

$$S_n = a_1 (1 - r^n)/(1 - r)."$$

Lemma 3: "Let $\sigma(n)$ denote the sum-of-divisors function, where n is a positive integer, and let H_n denote the n th harmonic number where $H_n := \sum_{k=1}^n (1/k)$. Then the inequality

$$\sigma(n) < H_n + \ln(H_n) \exp(H_n)$$

holds for all integer $n > 1$."

Unfortunately we do not have a partial sum formula for H_n , so we must find the contrapositive statement of Lemma 3 to prove it. We know that as $n \rightarrow \infty$, H_n diverges since it is a divergent p -series. But we are not evaluating an infinite summation, we are evaluating a finite sum since n is a finite number. Thus, we can subtract some terms χ from H_n :

$$H_n = 1 + (1/2) + (1/3) + (1/4) + (1/5) + \dots + (1/n), \quad (\text{no.45})$$

$$\Rightarrow H_n - \chi = (1/2) + (1/4) + (1/8) + (1/16) + \dots \quad (\text{no.46})$$

As we let $n \rightarrow \infty$, the terms from the line (no.46) approach a convergent geometric series with $r = (1/2)$ and $a_1 = (1/2)$. We shall denote this $S_n = H_n - \chi$. Since S_n is a convergent series that converges to 1 as $n \rightarrow \infty$, and H_n is a divergent series as $n \rightarrow \infty$, we can formulate the contrapositive statement to Lemma 1. This is because when looking at the interval of the natural numbers $[1, \infty)$, ∞ is the converse of 1 in the context of this problem. In particular, for the contrapositive statement to Lemma 1 we have that the inequality

$$\sigma(n) > S_n + \ln(S_n)\exp(S_n) \quad (\text{no.47})$$

holds for all positive integers $n > 1$. For the value ($n = 2$), we have

$$\sigma(n) > \frac{1}{2} [1 - (1/2)^n] / [1 - (1/2)] + \ln(S_n)\exp(S_n), \quad (\text{no.48})$$

$$\Rightarrow \sigma(n) > 1 - (1/2)^n + \ln(1 - (1/2)^n)\exp(1 - (1/2)^n), \quad (\text{no.49})$$

$$\Rightarrow \sigma(2) > 1 - (1/2)^2 + \ln(1 - (1/2)^2)\exp(1 - (1/2)^2), \quad (\text{no.50})$$

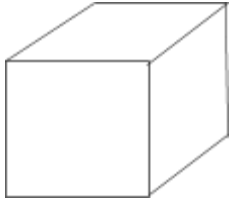
$$\Rightarrow 3 > 1 > (3/4) + \ln(3/4)\exp(3/4). \quad (\text{no.51})$$

The statement from the line (no.51) is true so the inequality from the line (no.47) holds for ($n = 2$). Let $f : \mathbf{N} \rightarrow \mathbf{N}$ and define $f(n) := 1 - (1/2)^n + \ln(1 - (1/2)^n)\exp(1 - (1/2)^n)$ from the line (no.49). We know that f is a strictly increasing function on the interval $[2, \infty)$, and

$1 > f(2) = (3/4) + \ln(3/4)\exp(3/4)$. Also, as $n \rightarrow \infty$, $f(n) \rightarrow 1$. This means that $1 > f(n)$ on the interval $[2, \infty)$. But $\sigma(n) > 1$ on the interval $[2, \infty)$. Thus $\sigma(n) > f(n)$ on the interval $[2, \infty)$.

Therefore Lemma 3 holds by its contrapositive statement.

Q.E.D.



Remark 1: This marks the end of what I formally submitted to the Teddy Rocks Maths Essay Competition. Everything that is included beyond this point is optional reading material on some random results.

Lemma 4: “Let $Q(n)$ denote Hofstadter’s Q -sequence as defined in Definition 4. Then, the limit $\lim_{n \rightarrow \infty} [Q(n)/n] = 1/2$.”

From the arguments made in Lemma 1 and the lines (no.20) we have that for all integers $n \geq 5$: $n > Q(n-1)$ and $n-1 > Q(n-2)$. Where $n-1 > Q(n-2)$ is yielded from a shift in the index n by 1. This implies that $n-1 \geq Q(n-1)$ and $n-2 \geq Q(n-2)$. The reason we do a shift in the index n and not use the equation from the line (no.22) is because we would have a way to compare how large $Q(n-1)$ is to $Q(n-2)$. If we were to set $n-1 = Q(n-1)$ and $n-2 = Q(n-2)$, being the upper bounds of the terms $Q(n-1)$ and $Q(n-2)$ we would get

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)), \quad (\text{no.52})$$

$$= Q(n - (n-1)) + Q(n - (n-2)), \quad (\text{no.53})$$

$$= Q(1) + Q(2), \quad (\text{no.54})$$

$$= 1 + 1, \quad (\text{no.55})$$

$$= 2. \quad (\text{no.56})$$

This shows that $n-1 = Q(n-1)$ and $n-2 = Q(n-2)$ only when $Q(n) = Q(1) + Q(2)$. This only occurs for small values of n . We can then establish another upper bound for larger n by having $n-2 \geq Q(n-1)$ and $n-3 \geq Q(n-2)$. If we were to set $n-2 = Q(n-1)$ and $n-3 = Q(n-2)$, being the upper bounds of the terms $Q(n-1)$ and $Q(n-2)$ we would get

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)), \quad (\text{no.57})$$

$$= Q(n - (n-2)) + Q(n - (n-3)), \quad (\text{no.58})$$

$$= Q(2) + Q(3), \quad (\text{no.59})$$

$$= X + Y. \quad (\text{no.60})$$

From the lines (no.59) and (no.54) we know that as we decrease the upper bounds $n - a \geq Q(n - 1)$ and $n - (a + 1) \geq Q(n - 2)$ by some a term, the inputs of X and Y increase. However, we know that there is a restriction to how large a can be since if we were to set $a = n$ we would get $n - n = 0 \geq Q(n - 1)$ and $n - (a + 1) = -1 \geq Q(n - 2)$. We know that $-1 \geq Q(n - 2)$ is a false statement since the terms of the sequence are nonnegative. Also, as a grows larger n grows larger since the inputs of X and Y increase, as seen on the lines (no.59) and (no.54), which allows us to describe the asymptotic behavior of $Q(n)$. If we were to set $a = n/2$ we would get $n - n/2 = n/2 \geq Q(n - 1)$ and $n - (n/2 + 1) = n/2 - 1 \geq Q(n - 2)$. If we were to set $n/2 = Q(n - 1)$ and $n/2 - 1 = Q(n - 2)$, being the upper bounds of the terms $Q(n - 1)$ and $Q(n - 2)$ we would get

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2)), \quad (\text{no.61})$$

$$= Q(n - n/2) + Q(n - (n/2 - 1)), \quad (\text{no.62})$$

$$= Q(n/2) + Q[(n + 2)/2], \quad (\text{no.63})$$

$$= X + Y. \quad (\text{no.64})$$

We don't know whether $Q(n)$ is defined at n when $a = n/2$, which is what we have to figure out. We know that by looking at the input of the X term from the line (no.63) and (no.64) we have $IQ(n/2) = n/2$. We know that we can evaluate this at n by having $Q(n)$ and performing a substitution on the input to give us $Q(n/2)$. We can do this because we know that if the sequence is defined at some n_0 , then it is defined for all $n < n_0$; in other words we should be able to define $Q(n)$ from X and Y . Looking at the input of the Y term from the line (no.63) and (no.64) we have $IQ[(n + 2)/2] = (n + 2)/2$. We know that we cannot evaluate this by performing a substitution on the input of $Q(n + 2)$ at n by the definition of a recursive sequence. However, since the input of Y is dependent upon $Q(n - 2)$ and we can define $Q(n - 1)$, then we must be able to define $Q(n - 2)$ by construction of recursive sequences. This means that $n/2 \geq Q(n - 1)$ and $n/2 - 1 \geq Q(n - 2)$ is true for large values of n .

If we were to set $a = n/2 + 1$ to modify the upper bounds of $Q(n - 1)$ and $Q(n - 2)$ then we would get $n - (n/2 + 1) = n/2 - 1 \geq Q(n - 1)$ and $n - (n/2 + 1 + 1) = n/2 - 2 \geq Q(n - 2)$. If we were to set $n/2 - 1 = Q(n - 1)$ and $n/2 - 2 = Q(n - 2)$, being the upper bounds of the terms $Q(n - 1)$ and $Q(n - 2)$ we would get

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2)), \quad (\text{no.65})$$

$$= Q(n - (n/2 - 1)) + Q(n - (n/2 - 2)), \quad (\text{no.66})$$

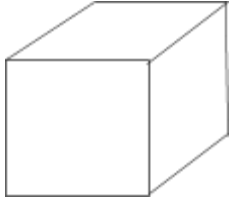
$$= Q[(n + 1)/2] + Q[(n + 4)/2], \quad (\text{no.67})$$

$$= X + Y. \quad (\text{no.68})$$

Again we don't know whether $Q(n)$ is defined at n when $a = n/2 + 1$. We know that by looking at the input of the X term from the lines (no.67) and (no.68) we have $I(X) = (n + 1)/2$. We

know that we cannot evaluate this by performing a substitution on the input of $Q(n+1)$ at n by definition of a recursive sequence. Similar reasoning is used for the input of the Y term. Thus we know that $a \neq n/2 + 1$, hence we have that $n/2 - 1 < Q(n-1)$ and $n/2 - 2 < Q(n-2)$, which implies $n/2 \sim Q(n-1)$ and $n/2 - 1 \sim Q(n-2)$ as $n \rightarrow \infty$ by the squeeze theorem. It follows that $(n+1)/2 \sim Q(n)$, but since constant additive terms are insignificant we get $n/2 \sim Q(n) \Rightarrow 1/2 \sim Q(n)/n$. This implies that $\lim_{n \rightarrow \infty} [Q(n)/n] = 1/2$. Therefore, Lemma 4 holds by Lemma 1 and definition of recursive sequences.

Q.E.D.



Remark 2: I just typed up the optional reading today, so I apologize if I made a mistake. I will have to check this again later.

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