Two opposite games involving golden ratio

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The golden ratio $\phi = 1.618...$ appears in many non-mathematical settings, like in some plants and architecture, and we often find this fascinating because mathematics shows up in some unexpected situations. The golden ratio, although irrational, also unexpectedly appears in two seemingly opposite games which are discrete in nature.

1 The two games

Suppose there are two piles of stones, and you are playing against an opponent. You and your opponent can count the number of stones in each pile beforehand to determine the winning strategy. You and your opponent take turns to remove some stones from the piles, but you can only do one of the following:

(i) Remove any number of stones in only one pile
(ii) Remove the same number of stones in both piles

You **cannot** skip your turn. The game continues until no more stones left in both piles. There are two ways to determine who win the game, which we will call *Method A* and *Method B* as follows:

Method A: the person who removed the last rock **wins**
Method B: the person who removed the last rock **loses**

What would be the winning strategy in either method? Are the winning strategies for *method A* different from *method B*? Does the strategy change depending on whether you are the first one to remove stones from the piles? Or should you actually "kindly invite" your opponent to take the first turn? If so, does that depend on the initial numbers of the rock in either pile? In both games, golden ratio ϕ is involved, and strangely, the winning strategies for the games using either method to determine the winner is exactly the same (with very slight caveat).

2 Building up the winning strategy

2.1 Definition of winning pairs

The road to building winning strategy for this game looks like this: we create sets of "*winning pairs*", so that in each turn, you should always aim to remove stones so that the remaining numbers of stones match these *winning pairs*. For example, if one of our *winning pairs* is $(9, 15)$, then we should aim to remove some number of stones so that one pile has 9 stones, and the other 15 stones. To construct our sets of *winning pairs*, we should have the following constraints:

(1) The numbers in different *winning pairs* should not repeat

The rationale behind this is simple: let's say there are two *winning pairs*: $(10, 14)$ and $(10, 18)$. If one pile has exactly 10 stones, and the other pile has 20 stones, how many stones should you remove from the second pile? To avoid ambiguity, we don't repeat numbers in the *winning pairs*. In addition, if you leave one pile having 10 stones, the other 18 stones, your opponent can easily remove 4 stones from the second pile to make him/her winning.

(2) The difference between the numbers in the *winning pairs* should not repeat

The rationale behind this is also similar, but instead of tackling rule (i), we tackle rule (ii). Using the *winning pair* $(9, 15)$ as an example, if we know that the difference in numbers of stones in the two piles is 6, then we can remove stones according to rule (ii) to arrive at $(9, 15)$. If $(10, 16)$ is also a *winning pair*, then we don't know how many stones to remove from each pile; and also if you leave one pile having 10 stones, the other 16 works, your opponent can easily remove 1 rock from each pile to make him/her winning.

Note that constraint (1) does not rule out the possibility of the same *winning pair* with repeated numbers like $(2, 2)$, but according to constraint (2), we have at most 1 of these. The best-case scenario is that all possible differences between the numbers in the *winning pairs* occur exactly once; and that all natural numbers also occur exactly once to allow us to win in all possible scenarios.

2.2 Constructing the winning pairs

We first tackle the case where the game is run according to *Method A*.

By *Method A*, $(0, 0)$ is obviously a *winning pair:* if you can remove all stones, you win!

Now notice that $(1, 2)$ is also a *winning pair*. If you can leave one pile of rock to have exactly 1 rock, the other having exactly 2, then no matter what the opponents do, you can always remove all the remaining rock and win the game.

The number 3 has not appeared yet. The differences between the numbers in the *winning pairs* right now only include 0 and 1, so the next difference should be 2. So maybe $(3, 5)$ is a *winning pair*? After some more thoughts, we can verify that it is indeed a *winning pair*: whatever your opponents do, you can always reduce to the *winning pair* $(1, 2)$ or win directly.

In general, we observe which number has not appeared on the list of *winning pairs*, say the number is A_n; then observe which difference has not appeared yet, say d, then $(A_n, A_n + d)$ should be the next *winning pair*. The table of *winning pairs* constructed will be as follows:

n	A_n	B_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
...	...	...

Table 1: ***Winning pairs*** **for** ***Method A***

Here, n is the index I give to the *winning pairs*, e.g. the third *winning pair* is $(4, 7)$, and A_n and B_n are the actual numbers in the *winning pairs*. You can check that these are really *winning pairs* - whenever you leave the numbers of stones in the piles follow (A_n, B_n), no matter what the opponents do, you can always reduce to previous cases.

For *Method B*, we construct the table of the winning pairs like what we did before, and here it is:

n	A_n	B_n
0	0	1
1	2	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
...	...	...

Table 2: ***Winning pairs*** **for** ***Method B***

Compare the two tables. Apart from the first two rows, they are identical. These two opposite methods for determining who the winner is yield the same winning strategy for the most part. Amd after generating these *winning* pairs, we can clearly state the winning strategy:

If the numbers of stones exactly match one of the *winning pairs*, kindly invite your opponent to start first. If not, please don't be too kind and let yourself play first.

Observe the difference in numbers of stones between the two piles, say d. Can you remove same number of stones from both piles so that it reaches the *winning pair* (A_d, B_d)? If yes, then do it.

If this is not possible, suppose the initial number of stones in the piles is (a, b), then look through the table to find where a and b are located. You must be able to remove stones from only one pile to match the *winning pair.*

For example, suppose the starting numbers are $(4, 10)$. This does not match any *winning pair*, so we need to play first.

The difference in numbers is 6, and look at $(A_6, B_6) = (9, 15)$. We cannot remove stones from the initial configuration of $(4, 10)$ to $(9, 15)$ obviously.

When we look at the table, 4 appears in the third *winning pair*, and 10 appears in the 4^{th} *winning pair*, but we clearly cannot remove stones from the initial configuration of $(4, 10)$ to $(A_4, B_4) = (6, 10)$; so the option we are left with is the thrid *winning pair* $(4, 7)$, i.e. we remove 3 stones from pile of 10 stones.

The question now is, are there any **efficient** ways to generate these *winning pairs*?

3 Relationship with golden ratio

Unexpectedly, apart from the first couple of *winning pairs*, we have

$$A_n = [n\phi] \tag{1}$$

$$B_n = \left[n\phi^2\right], \tag{2}$$

where $[x]$ is the integral part of x. For example, $A_7 = [7\phi] = [11.32...] = 11$, $B_7 = [7\phi^2] = [18.32...] = 18$.

The following will be a bit more mathematical for explaining why golden ratio appears, but it does not involve any complicated calculations.

Let's recall the defining property of the golden ratio ϕ. It satisfies

$$\phi^2 - \phi - 1 = 0. \tag{3}$$

So we have

$$\begin{aligned} B_n &= [n\phi^2] \\ &= [n(1+\phi)] \\ &= [n + n\phi] \\ &= n + [n\phi] \quad \text{(since } n \text{ is an integer)} \\ &= n + A_n, \end{aligned}$$

so the difference between the numbers in the *winning pair* is always n, the index of the *winning pair.* So A_n and B_n generated this way will always have different differences. But how do we know that the generated A_n and B_n only appear once in the table of *winning pairs* (except for the first couple)?

To prove this, by dividing both sides of (3) by ϕ^2, we get

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1. \tag{4}$$

Suppose for a contradiction that $[m\phi] = [n\phi^2] = p$ for some integers m, n and p, i.e. some numbers appear twice (or more) in the table of *winning pairs*. Then we have

$$p < m\phi < p + 1 \tag{5}$$

$$p < n\phi^2 < p + 1. \tag{6}$$

Dividing (5) by ϕ on all sides, and dividing (6) by ϕ^2 on all sides, we get

$$\frac{p}{\phi} < m < \frac{p+1}{\phi} \tag{7}$$

$$\frac{p}{\phi^2} < n < \frac{p+1}{\phi^2}. \tag{8}$$

Adding up (7) and (8), we get

$$p\left(\frac{1}{\phi} + \frac{1}{\phi^2}\right) < m + n < (p+1)\left(\frac{1}{\phi} + \frac{1}{\phi^2}\right). \tag{9}$$

Combining with the result (4), we finally obtain

$$p < m + n < p + 1. \tag{10}$$

However, since m and n are both integers, the sum of them is also an integer, and cannot be between two consecutive integers, i.e. a contradiction. Hence, all natural numbers can appear in the table at most once.

To show that all natural numbers will appear in the table, we run a similar argument as above: suppose for a contradiction that there is an integer p not in any *winning pairs*. This means that $p > m\phi$ and $p + 1 < (m+1)\phi$ for some integer m (i.e. when we take the integral part of $m\phi$ and $(m+1)\phi$, we cannot hit p), and similarly, $p > n\phi^2$ and $p + 1 < (n+1)\phi^2$, for some integer n. Using a similar trick as above, i.e. dividing these inequalities with ϕ by ϕ, and those with ϕ^2 by ϕ^2, we will get these four inequalities:

$$m < \frac{p}{\phi} \tag{11}$$

$$\frac{p+1}{\phi} < m + 1 \tag{12}$$

$$n < \frac{p}{\phi^2} \tag{13}$$

$$\frac{p+1}{\phi^2} < n + 1. \tag{14}$$

By adding (11) and (13) together; (12) and (14) together, we have

$$m + n < p\left(\frac{1}{\phi} + \frac{1}{\phi^2}\right) \tag{15}$$

$$(p+1)\left(\frac{1}{\phi} + \frac{1}{\phi^2}\right) < m + n + 2, \tag{16}$$

and again we can apply (4) to get

$$m + n < p \tag{17}$$

$$p + 1 < m + n + 2, \tag{18}$$

which means $m + n < p < m + n + 1$, which is again impossible for an integer to trap between two consecutive integers. Hence, all natural numbers will be in one of the *winning pairs* at some point.

This means A_n and B_n generated using the golden ratio fulfils our criteria for the *winning pairs* exactly!

4 Closing remarks

This is a variant of the Wythoff's game, and this game of removing stones was actually played by Chinese people. But this simple game only involving some discrete number of stones somehow involves the golden ratio! On top of it, the fact that the winning strategy stays largely the same for the two methods for determining who wins is also fascinating. If this relationship sounds very surprising to you, this exactly proves that there may be many more intriguing and unexpected connections within mathematics, waiting for you to discover!