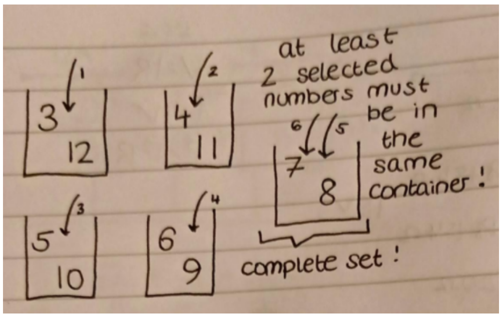
A very hairy principle

One of the most engaging components of maths, in my opinion, is its difference to all other subjects. A mathematical proof is a very different thing to a scientific proof. In scientific research, evidence is the foundation on which theories and explanations for certain phenomena are built. Observations and practical results are gathered, ideas and theories are suggested, all of which point towards an answer - but there is no way of saying for certain that this conclusion is correct.

Maths on the other hand, is certain. A proof in maths cannot be argued with, Pythagoras's theorem is not an opinion, but a fundamental law that has been proven beyond any doubt and must be obeyed (yes, even in hyperbolic geometry!). It is for this reason, that I find mathematical theories like pigeonhole principle so interesting - the way that they are able to justify answers to questions beyond any contradiction or argument, without the need for unholy quantities of evidence and research.

The pigeonhole principle states that if n items are put into m containers, with there being more items than containers, then at least one container must contain more than one item. I would like to be able to generalise this statement into a mathematical equation, but the maths is so hairy and confusing that I decided to leave that detail to the interested reader.

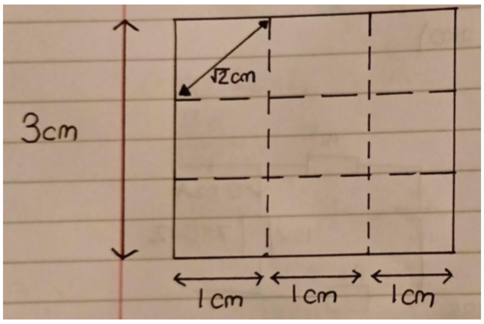


I will try to demonstrate the application of this principle through a simple example. Take the numbers 3,4,5,6,7,8,9,10,11 and 12. If I were to select six integers from this array, prove that there must be 2 integers which sum to 15. The best place to start with this question is to notice that there are 5 pairs that sum to 15: 3 + 12, 4 + 11, 5 + 10 and so on. Imagine that, once a number has been drawn, it is placed into one of five containers. Each container will be labelled with one of the pairs of numbers that sum to 15. We are allowed to draw out six numbers. This means that these six integers will need to be sorted into the five containers - therefore one container must have 2 numbers in it. And there you have it, we are guaranteed to have at least a set of a pair of numbers summing to fifteen!

Now this sounds pretty easy - obvious even - but it can give rise to some pretty fun results. For example, if the population of London is greater than the maximum number of hairs present on a person's head then the principle requires there to be at least two people with the same number of hairs on their head. This makes sense if we consider each person in London having a different number of hairs, say the first person has zero, the next one, the next two and so on. If we continue this up until the maximum hairs we will find that we do not have enough different degrees of hairiness to account for everyone. Therefore, at least two people must have the same number of hairs on their head. What we have done is created - in essence - a pigeon hole for each possible number of hairs, and have found that we have too many people to give each one of them their own hole. Now going back to my original point, there is no large scale investigation commencing in London to count the number of hairs on every person's head. Nor will there ever be. The mathematical proof provided by this principle has made this sort of survey completely redundant.

Pigeonhole principle is commonly called Dirichlet's box principle after Peter Dirichlet who developed the idea in 1834. Initially, Dirichlet used the metaphor of distributing pearls among drawers when explaining the principle. Now we use the rather old fashioned 'pigeonhole', in the sense of an open space in a wall for storing papers. When the term morphed into pigeonhole principle is hazy, but the quirk of the name has meant that the literal translation of the word 'pigeon hole' has found its way into the German translation 'Taubenschlagprizip' - meaning dovecote principle. In all honesty, I prefer the idea of actual pigeons nesting in boxes when visualising the principle.

My initial issue with pigeonhole principle was that I just could not see the point in it. Like a lot of pure maths, it seemed to have no connections to real life and it seemed mind bending to me that so much time and effort had been put into something so abstract. However, what I think I was missing was that where pure maths discovers, application soon follows. Take group theory, which has an integral role in quantum mechanics. Or topology - which has formed a foundation from which string theory was built. Even pigeonhole principle. During a recent maths club meeting, we discussed the applications of this theory in social distancing for coronavirus. If there is a floor or area 10m^2, this can be broken down into square blocks 1m by 1m. If each person is to have their own square block, we know that if more than 10 people are in this area then there must be at least 2 people in one meter squared block. This can be used to work out how many people can safely fit into a certain room, without people having to pop each other's social distancing bubble. All in all, what makes pure maths so exciting is that no-one really knows what problem they might be solving. It comes back to that fundamental part of being human, our need to invent and be creative before understanding and application kick in.



Another example of pigeonhole principle providing an elegant solution to a problem is with the points on a page question. The question goes: prove that if 10 points are placed on a three by three centimeter square, then two points must be less than or equal to root 2 centimeters apart. We know that we are able to split up a 3cm by 3cm piece of paper into nine 1cm by 1cm boxes, and we know that the diagonal of a one centimeter by one centimeter square is root 2 centimeters. If we divide up the paper in this way, we are left with nine 'pigeonholes'. If we are to place the 10 points among these boxes, we know that one 1cm by 1cm square will contain at least two points. The furthest away from one another these points can be is root 2 centimeters, therefore we can conclude that there will be two points less than or equal to root 2 centimeters apart.

The full list of beautiful problems involving this theory is too large to detail here, but I thought I would leave you with one of my favourites. At a business meeting, no one shakes their own hand and no one shakes another's hand more than once. Prove that there are two people who have shaken hands the same number of times. Let's first consider just five people. Somebody in this group could shake nobody's hand, one person's, two, three or four people's hands. Therefore there are five possibilities - or pigeon holes - that somebody can fall into. However, if one person shakes nobody's hand, then it is impossible for someone else to have shaken four people's hands. Because of this, we can say that shaking zero hands, and shaking four hands are mutually exclusive! This leaves us with only four possibilities. Four pigeon holes, five people. To write it algebraically, we would have n people and n - 1 categories. This means we can conclude that two people must have shaken the same number of hands.

In conclusion, I hope I have been able to pass on a portion of my interest in pigeon hole principle. It is the elegant and succinct nature of these ideas that make me enjoy maths to the extent that I do. I wish all areas of life could be explained and proven to the extent that mathematical proofs are capable of - although sadly I think that is a satisfaction reserved only for mathematicians.

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Sources
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https://www.youtube.com/watch?v=ROnetLvbl6M
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