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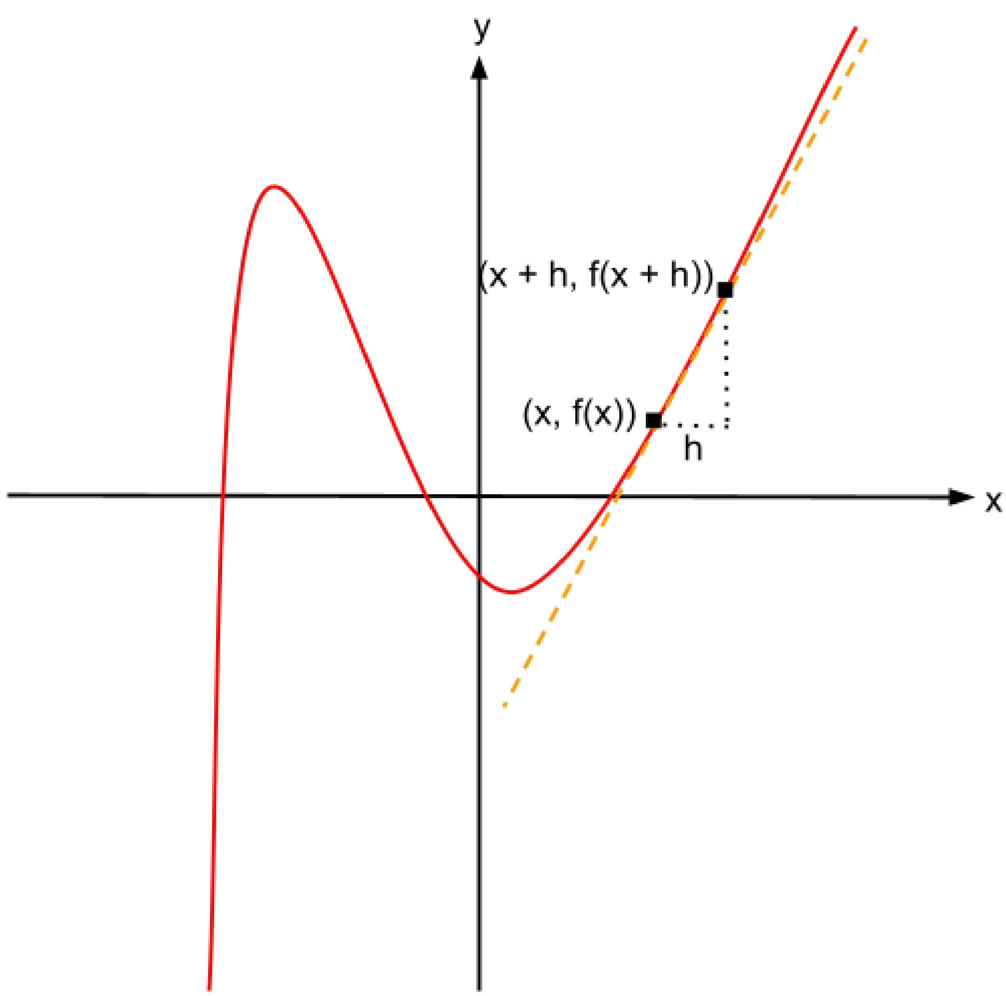
Differentiation Saved Our Lives

Differential calculus explores the rate of change of a variable with respect to another. The rate of change of a function can also be called its gradient, showing how steep a curve is. This rate is straightforward to calculate when the function is linear because the gradient is the difference in the ***y*** values between two points divided by the difference in the ***x*** values between two points. However, when the function is nonlinear, such as in a quadratic or cubic graph, you cannot find a specific gradient for it because the gradient changes across the graph. An expression can be found for the gradient of that curve at any point on it; this is called the derivative of a function.

When a function is nonlinear, its graph is curved and the way to find the gradient of a point on it is by drawing a tangent to the curve and finding its gradient. A formula for finding the derivative of a function can be found by using this same principle. Take the cubic graph below (red) for instance.



To find the gradient of this graph at any point, a tangent (orange) is drawn and two points on that tangent have been selected to find the gradient: ***(x, f(x))*** and ***(x + h, f(x + h))*** where h is the horizontal distance between the 2 x values. Using these coordinates, the gradient would be equal to:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

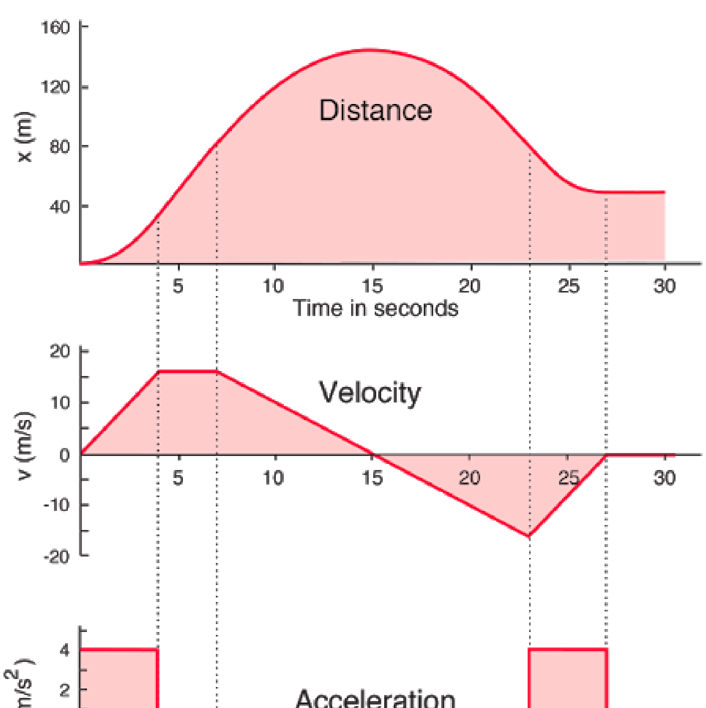
As the two points on the tangent get closer together, the gradient calculated with the new points gets closer to the gradient of the curved function. In other words, as ***h*** gets closer to 0, the gradient gets closer to the gradient function. For example, the first derivative of x^2 is:

$$\lim_{h \to 0} f'(x) = 2x$$

$$f'(x) = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x + h$$

The above formula is called the first derivative of a function, and the gradient of a curved function at any coordinate can be found by substituting the ***x*** value into the first derivative function.

Differentiation is applied to many aspects of life including in applied physics in the form of differential equations. Differential equations relate a function to its derivative and these can be used to show how that function changes over time. For example, the gradient of a displacement-time graph is equal to the velocity so the rate of change of displacement with respect to time (the first derivative of displacement) is equal to the velocity. Furthermore, the gradient of a velocity-time graph is equal to the acceleration, meaning the rate of change of velocity with respect to time (the first derivative of velocity) is equal to acceleration (as illustrated below).



The main reason I'm interested in differentiation is because of its prominent use recently in the COVID-19 pandemic. Differential equations have been of pivotal use to epidemiologists in understanding the rate of change of infection rates of the COVID-19 pandemic. Epidemiologists based more complicated mathematical models on the Susceptible - Infected - Removed model (SIR). Susceptibles (***S***) are people who are able to contract COVID-19, infected people (***I***) are people who currently have COVID-19, and the removed people (***R***) are people who have either recovered or died from COVID-19.

1) Susceptibles in terms of time = S(t).
2) Infected in terms of time = I(t).
3) Removed in terms of time = R(t).

Differential equations are made for each category with respect to time to predict how a fixed population is affected by COVID-19. The number of susceptible people decreases as a pandemic continues because more people get infected over time, meaning its differential equation will be negative. Susceptibles become infected with COVID-19 when they come into contact with infected people, this can be modelled as the susceptible people multiplied by the infected people: ***SI***. Only a proportion of these people will actually get infected with COVID-19, so naming this proportion ***a*** the susceptible differential equation becomes:

$$\frac{dS}{dt} = -aSI$$

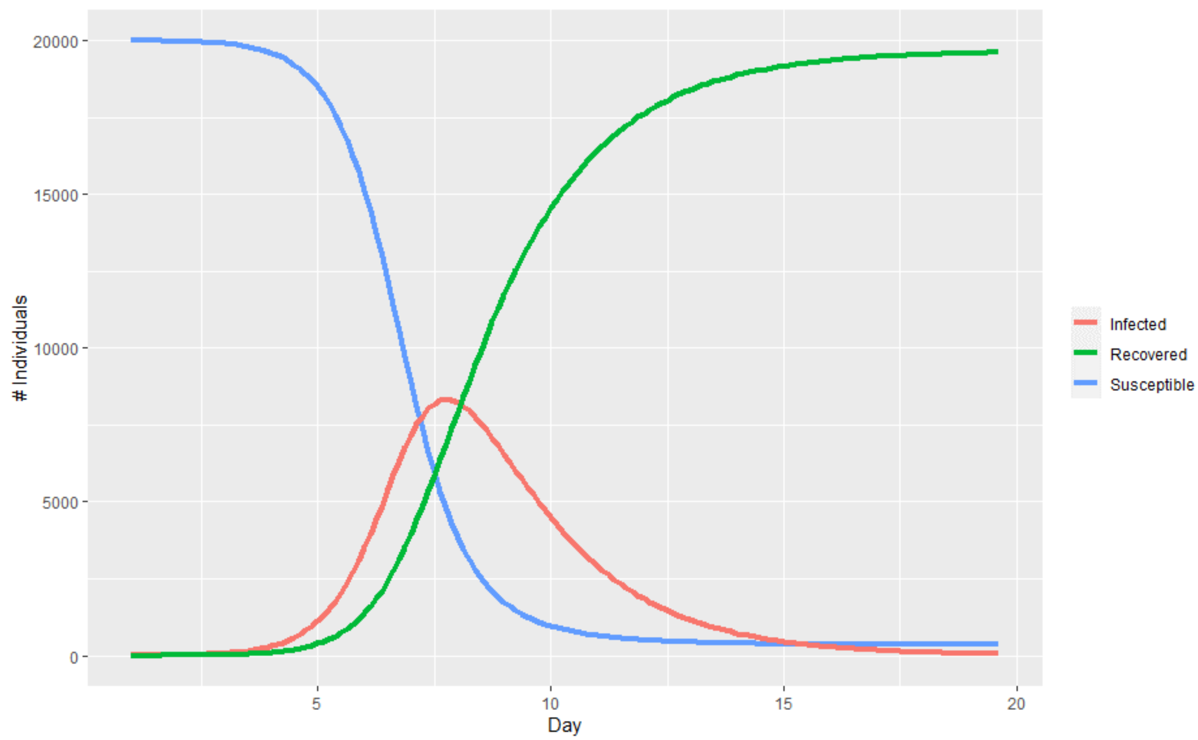
The people above leaving the susceptible category will be added to the infected category, meaning the number of infected people increases at the start of a pandemic. A different proportion of these infected people will be taken away from this category and added to the removed category as people either recover or die from COVID-19. Modelling this recovery rate as ***b***, the infected differential equation becomes:

$$\frac{dI}{dt} = aSI - bI$$

The removed category's differential equation is easier to create because the infected people taken away will just be added to this category: [1]

[1] T. Bazett, 'The MATH of Epidemics | Intro to the SIR Model', 2020, https://www.youtube.com/watch?v=Qrp40ck3WpI, (accessed 22 January 2021)

$$\frac{dR}{dt} = bI$$



Above is a graph showing how the SIR model models the basic changes to a population of 20,000 people over a 20-day epidemic. [2] In this, the susceptible population (blue) decreases and the recovered population (green) increases. The infected population (red) increases at the start when there are many susceptible people, showing the positive variable of its differential equation dominating. It then decreases when the susceptible population is smaller and more people are recovering, showing the negative variable to then dominate. The SIR differential equations can be used to predict the change of the three categories when policy-makers impose restrictions to try reducing the spread of COVID-19, such as social distancing or wearing masks.

More complex versions of the SIR model are being used to model COVID-19 as the SIR model contains many assumptions, the main one being that the population remains fixed throughout the pandemic. This variation of the SIR model takes into account the constant births and natural deaths that change the overall population over the course of an epidemic, called the SIR model with vital dynamics. This model assumes a constant birth rate (**μ**) which is equal to the death rate. This means the new terms with this variation will only include one constant. When considering birth rate, anyone in the total population (N) can give birth, regardless of what category they're in, so some amount (**μ**) of the total population needs to be added. When someone is born, they have not been in contact with COVID-19 so this model assumes they are born susceptible, so these births are only added to the susceptible

[2] B. Collins, 'Dynamic Modeling of Covid-19', *kx*, 2020, https://kx.com/blog/dynamic-modeling-of-covid-19/, (accessed 22 January 2021)

differential equation. The number of people passing away naturally is considered proportional to the number of people in each category so a proportion (**μ**) of each category is taken away in their differential equations. Adding these terms to the SIR differential equations gives: [3]

$$\frac{dS}{dt} = -aSI + \mu N - \mu S \quad \frac{dI}{dt} = aSI - bI - \mu I$$

$$\frac{dR}{dt} = bI - \mu R$$

This next variation of the SIR model is specific to COVID-19 as there is a significant delay period between the time of contracting the virus and symptoms showing, On 4th March 2020, the Scientific Advisory Group for Emergencies (SAGE) estimated this period between 1 and 11 days for the UK. [4] This rendered the SIR model less useful because many people who have been contracting COVID-19 show little to no symptoms as this virus is more severe on the elderly. [5] To combat this, the SEIR model has been made with a fourth category called exposed (***E***) for people who are capable of infecting others but display no symptoms themselves. The addition of this category means the proportion of susceptible people in contact with infected people have been exposed to COVID-19 instead of directly to the infected, meaning this proportion is added to the exposed category. Susceptible people are also able to contract COVID-19 when they come into contact with exposed people, which is modelled as susceptible people multiplied by exposed people. A new proportion (***d***) of these people are taken away from the susceptible category and added to the exposed category. Then a different proportion (***c***) of exposed people are taken away and added to the infected category. Adding these changes to the SIR model give the following differential equations:

$$\frac{dS}{dt} = -aSI - dSE \quad \frac{dE}{dt} = aSI + dSE - cE$$

$$\frac{dI}{dt} = cE - bI \quad \frac{dR}{dt} = bI$$

The third variation is currently very useful in understanding the rate of change of COVID-19 once vaccines are introduced to the population, encouraging herd immunity. The number of people

[3] T. Bazett, 'The MATH of Epidemics | Variants of the SIR Model', [online video], 2020, https://www.youtube.com/watch?v=f1a8JYAixXU, (accessed 4 February 2021)
[4] Scientific Advisory Group for Emergencies, 'COVID-19 compared with NSRA pandemic influenza planning assumptions (draft)', 4 March 2020', *Gov.uk*, 2020, https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/904538/S0641_COVID-19_compared_with_NSRA_2019_Pandemic_Influenza_planning_assumptions.pdf, (accessed 3 February 2021)
[5] National Center for Immunization and Respiratory Diseases (NCIRD), 'Older Adults and COVID-19', *CDC*, https://www.cdc.gov/coronavirus/2019-ncov/need-extra-precautions/older-adults.html, (accessed 15 January 2021)

getting vaccinated over a period of time is independent of any category as everyone from each category is just as likely to get vaccinated. This means the number of people getting vaccinated is an independent variable (***x***). The people who have gotten vaccinated are taken away from the susceptible category and moved directly to the removed category, giving these differential equations:

$$\frac{dS}{dt} = -aSI - x \qquad \frac{dI}{dt} = aSI - bI$$

$$\frac{dR}{dt} = bI + x$$

In conclusion, differential calculus is an "integral" mathematical topic which explores the rate of change of different variables. It has been used by epidemiologists to understand the rate of change of COVID-19 to populations across the world and to predict how these rates would change with different restrictions. Differential calculus should be considered as an essential element of mathematics that the wider population engage with, given its place and significance modelling the global pandemic. This essay has explored the use of differentiation in the SIR model and its variations to understand how useful and truly versatile differential calculus is to society.

Word Count: 1517

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