Applied Vector Calculus: The Divergence Theorem

"Mathematics is the language in which God has written the universe". This particular quote, from Italian astronomer Galileo Galilei, holds an importance to me that neither book nor video nor lecture could possibly achieve.

When I first heard this quote, it reminded me of the real purpose behind the field of mathematics (and especially mathematical physics): it is an eternal quest to understand the universe that surrounds us. Certainly, mathematics is a wonderfully vast and diverse subject, spanning from the purest algebra to the most abstract geometry and even further to the essential applied fields (economics, statistics and the like). Nevertheless, there comes a point when mathematics ceases to be merely elegant and conceptual, and becomes something far greater, far more all-encompassing, in my opinion at least.

As you can probably tell by now, I have a personal preference for using mathematics in applied, real-world scenarios; I believe that mathematics and physics have a uniquely intimate connection. Consequently, it will come as no surprise that the building blocks of mathematical physics, specifically the divergence theorem, will be the subject of this essay.

The divergence theorem (commonly referred to also as *Gauss's Theorem*), and indeed all of the myriad equations and functions associated with it, are arguably the perfect union between the worlds of pure mathematics and mathematical physics. It is so fundamental in fact, that it appears on the first page of several mathematical physics textbooks. Yet, the theorem itself is a description of a most trivial property of nature: the *law of conservation* of matter and energy.

In terms of vector calculus, divergence is a differential operator that helps to describe the net flux of a vector quantity over the boundary of an enclosed domain (this is usually used to model a physical quantity in the context of *potential theory*). Although this is a fairly straightforward property, this terminology may be rather daunting to a non-specialist, so I will attempt to describe it as fundamentally as I can.

For those who are unaware, a vector is a quantity that possesses both a size (magnitude) and a direction. A vector field is the result of assigning a vector to every point in a given subspace. If we were to sample the windspeed across the whole country at any one time, our vector field would map the gusts of wind as they travel across the land.

Of course, modelling fluid flow is not the only use for such a field. With regard to mathematical physics, a vector field is most commonly used to model some force or transfer of energy, whether that's gravitational attraction, motion as well as magnetic and electric fields. Indeed, many vector quantities can be represented as a vector field.

Perhaps the easiest way to visualise divergence is in the context of sources and sinks. A source is a point, or region, in a vector field at which the field seems to expand outward. If we imagine a faucet pouring water into a tray, it is clear that the water will spread out from that point since the water that is added to the system pushes out on its surroundings. Conversely, if I were to drill a hole in the bottom of the tray, the water will begin to move towards the hole until it pours out of the tray. A point at which the vector field converges is referred to as a sink (or a negative source mathematically).

This finally brings us to the concept of divergence. As mentioned before, the divergence of a vector field helps us to describe the net flux of the vector quantity over a domain boundary. Essentially, if we pick any miniscule region in a vector field, the divergence would determine the overall flux passing in and out of the local vicinity. Hence, in its most basic form, the divergence is defined as the extent to which a point (or small region) behaves as a source.

Grant Sanderson (3Blue1Brown) has a fascinating video on this topic, where he visualises these vector fields and demonstrates how the properties of divergence (and another operator, *curl*) relate to these fields. However, in this brief explanation, I want to focus on the mathematical aspect of divergence.

Although these are nice ways of visualising divergence, it is vague, very much so. We shall now define the proper divergence theorem using calculus and vectors. However, we must first lay down some basic vector algebra.

We can express a vector $\mathbf{v}$ (vectors are written here in bold) as a set of components (v_x, v_y, v_z) representing its projections onto the coordinate axes. In the case of a position vector $\mathbf{r}$, which starts at the origin, these components *are* the actual x, y, z values.

We can determine the magnitude (written as $|\mathbf{v}|$) of this vector using the *Pythagorean theorem*.

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

The notation $\mathbf{u} \cdot \mathbf{v}$, known as the *dot-product* between two vectors, is a combined measure of their alignment and magnitude, generally given as:

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z$$

Unit vector written as $\hat{\mathbf{v}}$ is a vector of magnitude one, defined as

$$\hat{\mathbf{v}} \equiv \frac{\mathbf{v}}{|\mathbf{v}|}$$

For our definition, we shall label our domain of interest as omega (Ω) and the domain boundary as gamma (Γ). At each point on Γ, one can construct a unit vector $\hat{\mathbf{n}}$ orthogonal to the local curvature of the boundary. This vector $\hat{\mathbf{n}}$ is called a unit outward normal, meaning it is a unit vector pointing out of the domain and perpendicular to the domain boundary. The direction of $\hat{\mathbf{n}}$ will, of course, vary across the boundary.

Now, let's talk about calculus. The physical properties of a system can be separated into two groups. If a quantity is dependent on the size of a system, it is an *extensive* quantity (volume and mass are two examples). However, if a quantity does not depend on the size of the system, it is an *intensive* quantity (these usually vary in space, like pressure and temperature).

Unlike extensive quantities, intensive quantities vary with space and, as a result, can be represented as a function of space. For example, if E represents the total energy inside our domain Ω, then it is an extensive quantity and is therefore independent of position. On the other hand, the local energy density inside Ω, which we can denote as e, varies from point to point: $e = e(\mathbf{r})$ (although $e(\mathbf{r})$ may be constant).

A spatial derivative describes how much an intensive quantity changes over an infinitesimally small distance in a given direction, say $\hat{\mathbf{x}}$, at every point $\mathbf{r}$:

$$q_x(\mathbf{r}) = \frac{\partial p(\mathbf{r})}{\partial x}$$

The x-derivative of p is a new function, q_x, which may again vary in space (i.e., with $\mathbf{r}$). Spatial derivatives of a scalar field, such as $p(\mathbf{r})$ (note: p is often called a *potential* because it represents some potential energy density), form a vector (force) field $\mathbf{q}(\mathbf{r})$:

$$\mathbf{q}(\mathbf{r}) = (q_x, q_y, q_z) = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) \equiv \nabla p$$

The quantity $\mathbf{q} = \nabla p$ is the *gradient* of p and its direction identifies the largest change of $p(\mathbf{r})$ in the vicinity of each point $\mathbf{r}$. One way to illustrate the gradient is to imagine a ball on a slope of a hill; the height z of each point $\mathbf{r}$ on the hill corresponds to some potential energy density $p(\mathbf{r})$ and $-\nabla p$ then points in the direction of the steepest decent, to where the ball would start rolling down due to the gravitational force acting on it.

From this, the spatial derivative of a vector field may be defined as the dot-product between the field and (the differential vector operator) ∇ as:

$$s(\mathbf{r}) = \nabla \cdot \mathbf{q}(\mathbf{r}) = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

The quantity $s = \nabla \cdot \mathbf{q}$ is the *divergence* of $\mathbf{q}$ and it represents sources/sinks (of matter, energy, etc.) in the vector field $\mathbf{q}$.

A spatial integral is kind of an opposite of the derivative: it accumulates the local values of some (scalar or vector) intensive quantity over a (generally curved) path, surface or volume. Here are examples of the three types of spatial integrals:

- line integral of force $\mathbf{F}(\mathbf{r})$ over path C (via dot-product with $d\mathbf{r}$) gives energy (or work):

$$E = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- surface integral of fluid velocity $\mathbf{v}(\mathbf{r})$ over surface area A, described by its unit outward normal $\hat{\mathbf{n}}$, gives volumetric flow rate of all fluid crossing the area:

$$Q = \iint_A \mathbf{v} \cdot \hat{\mathbf{n}}\, dA$$

- volume integral of energy density $e(\mathbf{r})$ over volume V gives the total energy E in there:

$$E = \iiint_V e\, dV$$

Now, at long last, we have all of the fundamentals needed to express the divergence theorem:

$$\iiint_\Omega \nabla \cdot \mathbf{q}\, d\omega = \oiint_\Gamma \mathbf{q} \cdot \hat{\mathbf{n}}\, d\gamma$$

This beautiful formula states that the cumulative divergence of a vector quantity $\mathbf{q}(\mathbf{r})$ throughout the domain Ω equals to the total flux of the same quantity across the boundary Γ fully enclosing Ω (hence the ellipse over the surface integral).
This is an extremely important mathematical statement, often used as a starting point for solving numerous problems in physics and engineering. It expresses conservation of mass, energy, momentum and of other quantities.

To illustrate the relationship, we can combine it with two other equations written previously:
1. $s = \nabla \cdot \mathbf{q}$, and 2. $Q = \iint_A \mathbf{v} \cdot \hat{\mathbf{n}}\, dA$, where $\mathbf{q}$ now replaces $\mathbf{v}$ and A is the entire area of the boundary Γ. This gives a much simpler statement

$$\iiint_\Omega s\, d\omega = Q$$

asserting that the total balance of all the internal sources and sinks s (e.g., of fluid in this case) over the domain volume equals to the net flow rate Q across the domain's boundary. If the total divergence cancels out, then $Q = 0$. That does not necessarily mean that $s(\mathbf{r}) = 0$ everywhere. One example of a non-zero s giving zero Q would be a point source and a point sink of the same strength cancelling each other out, as illustrated in the following three figures (generated in Wolfram Mathematica):

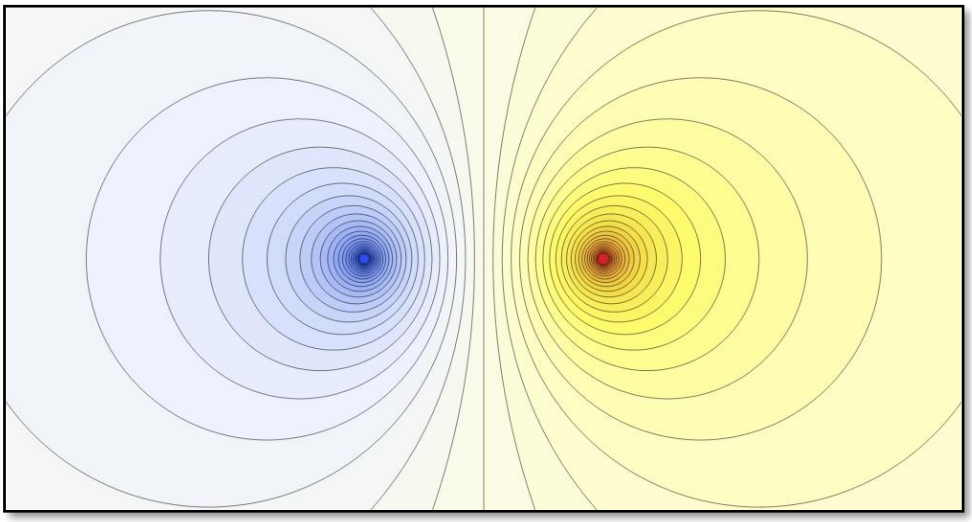

Figure 1: Scalar field of potential $p(\mathbf{r})$ around a point source and sink

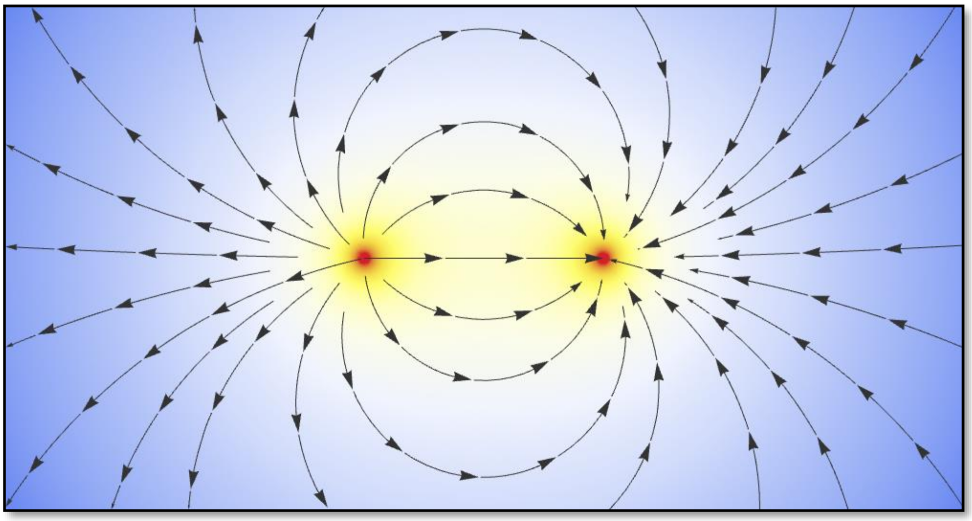

Figure 2: Vector field of force $\mathbf{q}(\mathbf{r}) = \nabla p$ induced by field $p(\mathbf{r})$ in Figure 1

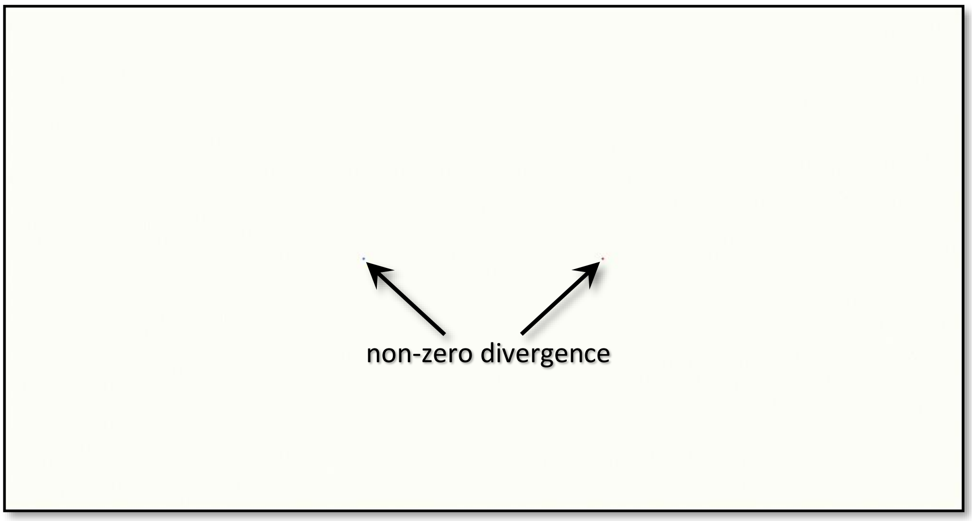

Figure 3: Scalar field of divergence $s(\mathbf{r}) = \nabla \cdot \mathbf{q}$; zero everywhere except the two points representing the source and sink in Figures 1 and 2

There are various types of internal sources. The obvious example, in this context of fluid flow, could be a pipe connected to some point inside the domain, pumping in fluid not accounted for in the surface integral. A less intuitive type of internal source, causing $Q \neq 0$, is an accumulation of compressible fluid shortly after some pressure changes. After a much longer time, however, the disturbances settle down and the system reaches a *steady state*, yielding $Q = 0$. The so-called *continuity equation* describes the *transient state* and may be expressed as *Poisson's equation*

$$\nabla \cdot \mathbf{q} = \nabla \cdot \nabla p = s$$

which simplifies to *Laplace's equation* if there are no internal sources (satisfied by so-called *harmonic functions*):

$$\nabla \cdot \nabla p = 0$$

These two fundamental equations, accompanied by some prescribed *boundary conditions*, may be viewed as the differential forms of the divergence theorem. One way to solve Poisson's equation (or a related *diffusion equation*) is to apply free-space *Green's function* $G(\mathbf{r}, \mathbf{r}')$, which solves the equation with s replaced by the *Dirac delta* function δ

$$\nabla \cdot \nabla G = \delta(\mathbf{r} - \mathbf{r}')$$

where δ represents a point source of unit strength at $\mathbf{r}'$. The solution process often involves *Green's identities* which are corollaries of the divergence theorem. It is interesting to note that the Green's first identity, for example, is a generalisation of the integration by parts to 3D space.

As a final note, let's take a moment to examine the equally rich history of this theorem. The idea of divergence was first formulated by Joseph-Louis Lagrange (French) in 1762. It was later proven for special cases by none other than Carl Friedrich Gauss (German) in the year of 1813. It wasn't until 1826, when Mikhail Vasilyevich Ostrogradsky (Russian) proved it as a general theorem. This is why the theorem is often referred to as Gauss' theorem in western countries and Ostrogradsky-Gauss theorem in non-western countries. Other special cases were proven 15 years later by George Green (English), who derived his three identities from the divergence theorem.

The point to be made here, is that this theorem is the result of many years of international collaboration, between nations that were by no means the strongest of allies. However, despite political tensions, in many ways these nations were united through mathematics. Now, if that isn't proof that mathematics is beautiful, I don't know what is!

I hope those who have read this far have gained a better understanding of mathematical physics outside of the classroom. Looking back on my education, I feel as though what I was taught didn't do these topics justice: a few formulae that were supposed to describe the world, almost feels meaningless in comparison. I truly mean it when I say that vector calculus changed the way I see the world, and I can only hope those who have read this can now see the world differently too.

by Tom Pecher