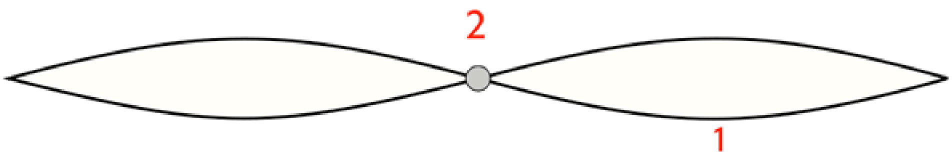
Is there a Relationship Between Maths and Music?

Pythagoras once said, 'there is geometry in the humming of the strings, there is music in the spacing of the spheres.' It may seem unlikely, but there is truth to be found in his words; Maths and Music are related. When most people hear the word 'Maths', the first thing that comes to mind is not normally artistic - surely Maths is based on logic and Music is based on emotion? But what if I told you that the beauty of the world around us wouldn't exist were it not for Maths? Think of your favourite pieces of music - have you ever considered that they may be embedded with mathematical patterns that subconsciously increase your enjoyment when listening to them? In fact, music as we know it can be accredited to the ideas of prominent mathematicians.

$a^2 + b^2 = c^2$. It is perhaps the most well-known mathematical formula to date, created by Pythagoras during the ancient Greek era. But Pythagoras did far more than torment GCSE students, he also made significant contributions to early instrumental tuning.

The Greeks were the first to link Maths and Music. Pythagoras and his followers formed a cult where they believed that numbers were sacred; they wanted to quantify everything, particularly musical intervals.

Over the last few thousand years, three different systems of instrumental tuning have been discovered, all of which are based on mathematical ratios. The Pythagorean tuning system was discovered in 6 BC when Pythagoras noticed that plucking strings at various lengths produced different pitches; from this, he deduced ratios between notes. As this diagram shows, if you were to stop a string halfway along, it would produce the ratio 2:1. Some ratios between notes produce consonant (satisfying) sounds; a perfect 5^{th} is in the ratio 3:2 and is an example of 'perfect consonance'.



He decided to base his entire system of tuning on perfect fifths. The Pythagorean scale of C major can be constructed using simple multiplication and division:

261 x 3/2 = 392 Hz

This is the note G on the Pythagorean scale.

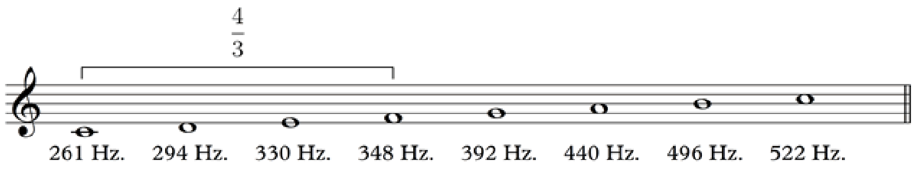
392 ÷ 4/3 = 294 Hz (D)

294 x 3/2 = 440 Hz (A)

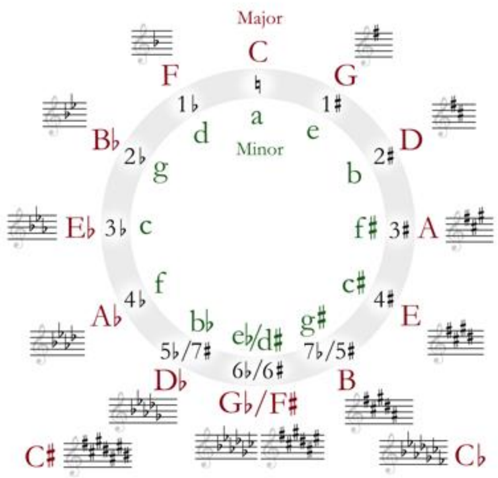
440 ÷ 4/3 = 330 Hz (E)

330 x 3/2 = 496 Hz (B)

261 x 4/3 = 348 Hz (F)



Pythagoras was able to define that an octave consisted of 12 semitones, creating the Pythagorean circle, which developed into the circle of fifths after revisions were made by Nikolay Diletsky in the 1670s, and Johann David Heinichen in 1728. The circle of fifths is a musical device which represents the key signatures of each scale and how they relate to each other. When read anticlockwise, you get the sequence of flat keys and when read clockwise, you get the sequence of sharp keys. Much like the Pythagorean system of tuning, each step is a perfect 5th from the next.



During the Renaissance, the preference for more complex rhythms such as thirds and fifths uncovered inconveniences in the Pythagorean system of tuning: the ratio for major third was 81/64. Another system of tuning needed to be created to make these ratios simpler. Eventually, further irregularities were found. If you look at the 7 octaves on a piano, the frequency doubles with each octave, so you multiply the frequency of the first note by:

$$2^7 = 128$$

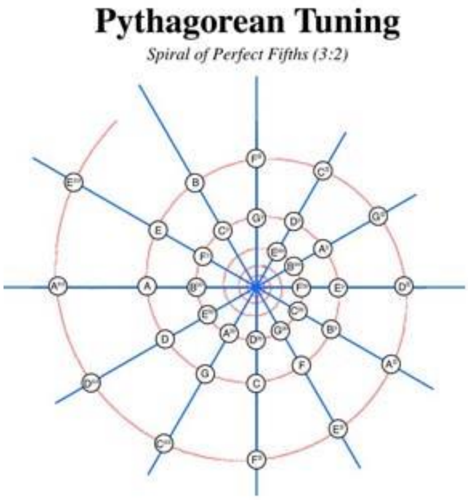
However, if you were to go all the way up a piano in fifths:

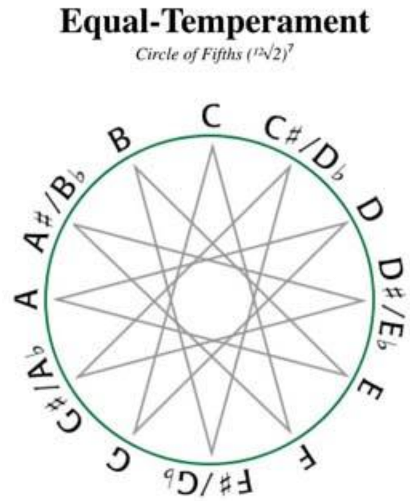
7 octaves = 12 fifths

$$(3/2)^{12} = 129.74633.$$

$$129.74633 \neq 128.$$

Therefore, it was found that Pythagorean system was flawed; this is known as the Pythagorean Comma. The circle of fifths cannot be used in this instance, because it would produce a spiral, not a closed circle.





To solve this problem, a system of just intonation was developed, whereby all the notes in a scale are related to each other by adding or subtracting rational numbers, namely the more consonant intervals. This included fifths (ratio 3:2) and thirds (5:4). During Medieval times, this didn't pose much of an issue because most music consisted of only single line melodies. However, as melodies became more complex, this system also proved to be flawed because the semitones within a scale were not at equal intervals from each other; this meant that the construction of a D major scale differed from a C major scale.

Today, we can play music in any key, on the same instrument, and a note will sound perfectly consonant with the same note several octaves higher. This is because equal temperament was developed in the 1700s. Harmonically adventurous composers needed to quickly switch between keys without re-tuning their instrument. For example, Bach's 24 Preludes and Fugues (1722) included all 24 major and minor keys which needed to be played in one sitting. Using equal temperament, the interval between each semitone is equal, but the ratios of intervals remain very similar to that of Pythagorean intervals:

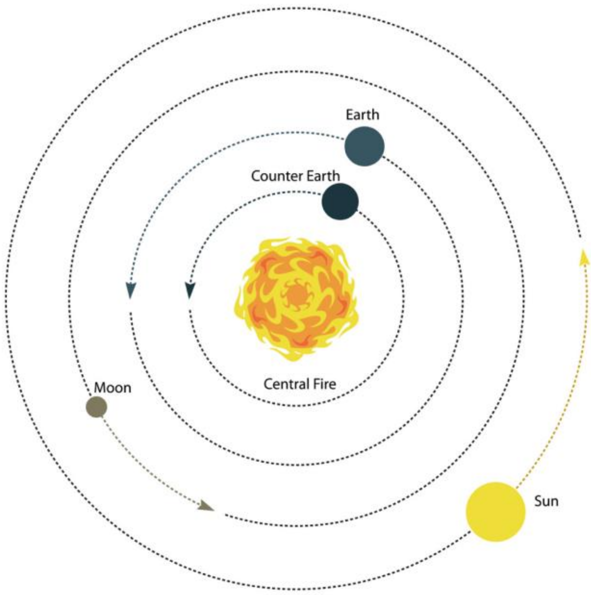
Interval	Equal Temperament	Pythagorean Tuning
Perfect fifth	1.498	1.5
Fourth	1.335	1.333
Major third	1.26	1.25

Pythagoras was not the only ancient Greek who linked Maths and Music. Plato founded the world's first university and is known for his identification of Platonic solids which he claimed were the basis for the whole universe: the tetrahedron, octahedron, icosahedron, cube and dodecahedron. The question arises as to why Plato would go on to say this:

'I would give the children Music, Physics and Philosophy, but the most important is Music, for in the patterns of the arts are the keys to all learning'.

Pythagoreans clearly thought music was an integral part of Mathematics. In ancient Greece, a Maths course included modules on number theory, geometry, astronomy, and music; a practice which was common in European culture until the Middle Ages.

Pythagoreans were convinced that all orbiting planets made harmonious sounds inaudible to the human ear: ‘harmony of the spheres’. This theory may seem laughable now, since sound cannot travel through a vacuum, but it significantly impacted both the way in which our current system of tuning was formed and our scientific understanding of the behaviour of planets. Pythagoreans believed that celestial objects orbited a ‘central fire’, as shown below, and that there was a geometric connection between the separation of these spheres and the harmonic lengths of strings. Plato saw a particular connection between Music and Astronomy; he believed both had strong mathematical undercurrents.



It took some time for this idea to be revisited and developed in order to accommodate modern beliefs. German mathematician Johannes Kepler (1571-1630) is famed for having discovered the three laws of planetary motion. As a young man, he was fascinated to learn of Plato’s beliefs. Building upon Nicolaus Copernicus’ theory that all planets revolve around the sun, in his 1597 book Mysterium Cosmographicum, he suggested that the distance between planets (Venus, Earth, Mars, Saturn, Jupiter, Mercury were the only ones known at the time) depended on Platonic solids. He then conducted a series of musical experiments, deciphering by ear which intervals he believed were the most consonant and whether these somehow related to the distance of the spheres. He eventually established a weak connection between the two and found that the ratio of Mars’ extreme orbital speeds measured in terms of angular motion across the sky, was about 3:2 (a perfect fifth). Although far-fetched and untrue, he published his findings in ‘Harmonices Mundi’ in 1619 and it was during his exploration of the harmonic structure of the cosmos that he discovered the third planetary law of motion. He was able to ascertain the time it took for a planet to revolve around the sun and thus deduced that planets orbited the sun in an elliptical rather than circular motion. In terms of musical development, Kepler was significant as he was one of the first to move away from the idea that scales could only be built around equal intervals, prompting further development of equal temperament.

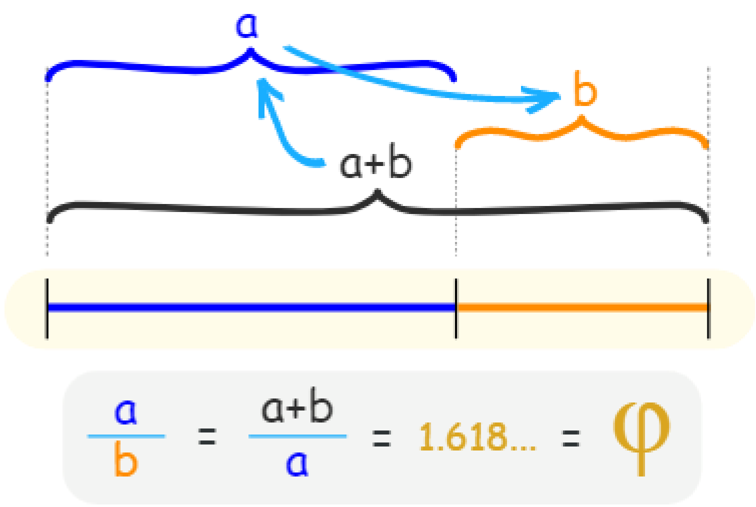
Do you think there is a perfect, almost divine way for things to look and sound? Leonardo Fibonacci was an Italian mathematician who wrote ‘Liber Abaci’, which introduced what is currently known as the Fibonacci sequence to the Western World from India. The sequence goes as follows and the next number in the sequence is calculated by adding the previous 2 terms:

1,1,2,3,5,8, 13...

This is important because it can be seen not only in the structure of some instruments but also in classical music compositions. On a piano, there are 13 keys in one octave, 8 white notes and 5 black notes. A triad (which is what arpeggios are based on) is created using the 1st, 3rd and 5th degrees of a scale. These are all numbers found in the Fibonacci sequence. Chopin's Prelude in C Major comprises of 34 bars with a climax occurring at bar 21. The first major chromatic event is in bar 13 and the opening statement lasts 13 bars. This forms a perfect Fibonacci sequence.

In the first movement of Mozart's Sonata in C Major, the climax of the piece happens at a place called the 'golden ratio' where:

Φ Phi=1.618



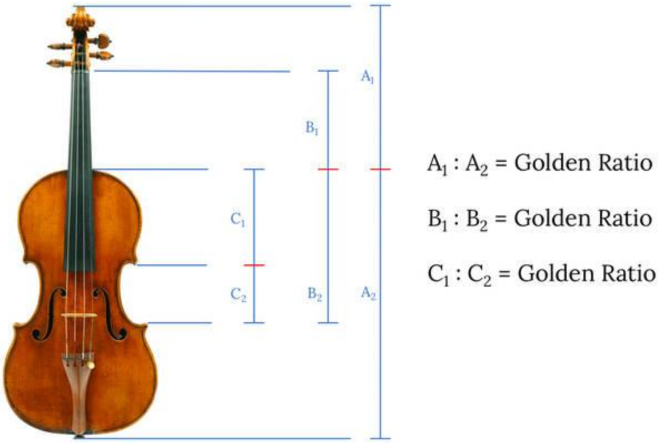
This is closely related to the Fibonacci sequence because if you divide the ratios of each term in the sequence, the product gets closer and closer to 1.618:

1/1 = 1	13/8 = 1.625
2/1 = 2	21/13 = 1.615
3/2 = 1.5	34/21 = 1.619
5/3 = 1.667	55/34 = 1.618
8/5 = 1.6	

The golden ratio can even be found in pop music. A recent song called 'Perfect Illusion" by Lady Gaga illustrates his point well. The entire piece lasts 179 seconds but a key change occurs at 111 seconds.

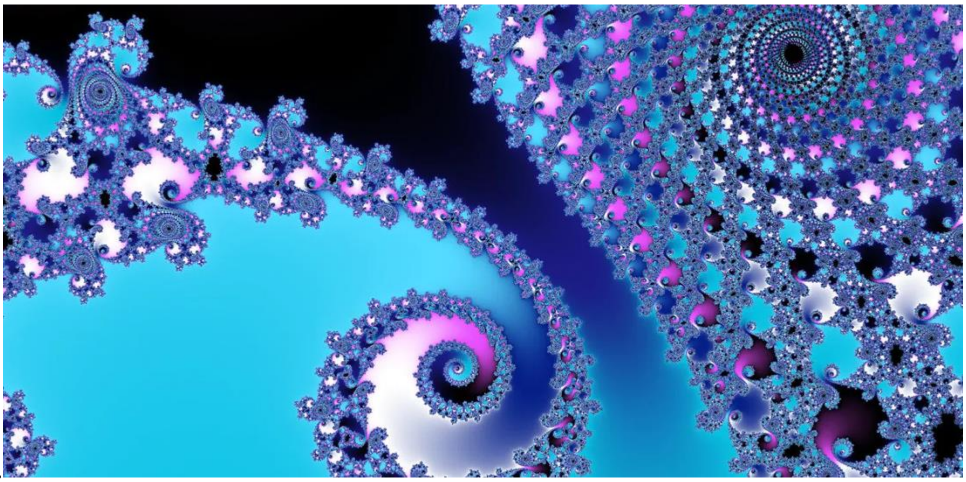
111 x 1.618 = 179.598

Antonio Stradivari was the mastermind behind what some consider to be the most beautiful sounding stringed instruments ever made. Rare and expensive, these instruments are among the most desirable for professional musicians. There is a mathematical reason as to why this is the case. If a Stradivarius violin is divided in relation to its main parts, the resulting ratios are equivalent to the golden Ratio.



The use of the Fibonacci sequence in the arts rose to greater prominence from the 1870s onwards, thanks to French mathematician Edouard Lucas' written works. This inspired French composer Debussy to use Fibonacci structures in his music, which would later be studied by Roy Howat. His extensive analysis of Debussy's 'La Mer' uncovered a 55 bar long introduction which can be broken down into 5 sections of 21, 8, 8, 5 and 13 bars in length. Yet another perfect Fibonacci sequence.

What if I told you that behind every tree, flower and cloud there was a complex mathematical structure? A fractal is an endless loop found in everything around us, it can be identified when a pattern looks the same at different scales. The idea that this could somehow be related to music seems impossible, but composer Kaija Saariaho wrote some of 'Nymphea' using a fractal music generator; she converted these infinite numbers into musical notation.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The concept of self-similarity in music can even be found in Bach's Cello Suite; the patterns of long and short notes in a small section reappear as patterns of long and short phrases on a larger scale. It has been suggested that the more fractals there are in music, the more likely we are to enjoy it.

It is unclear as to whether composers wrote their music with mathematics in mind, but the sheer number of pieces which display such concepts cannot be a mere coincidence. While there is no music in the spacing of the spheres, the very fundamentals of Music are derived from Mathematics. Mathematicians have made important contributions to Music in that it was Pythagoras who created the basis for how instruments are tuned today and Kepler who helped spark the development of equal temperament. Furthermore, the Fibonacci sequence and the golden ratio can be found not only in the manufacturing of instruments, but also in numerous compositions, old and new. I hope now that you think of Mathematics in a different light; it is far more than just numbers – it is the basis of beauty.

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