The Divine Proportion

Fundamentals of the golden ratio and its associations with geometry and the natural world

Plants have undergone evolution for millions of years in order to optimise their genetics for the environment which surrounds them. The process of photosynthesis is essential to provide glucose for the plants and it requires an abundance of sunlight to provide initial energy for the reaction. Thus, it is obvious that plants will have optimised this process over time, maximising the potential intake of sunlight. One of the ways in which this occurs is by arranging the leaves specifically to reduce overlap.

The law of divergence states that the angle formed by two consecutive leaves is constant and is known as the angle of divergence. Let's say that the angle of divergence is 360°, each leaf along the stem will be placed directly on top of one another so that the uppermost leaf blocks the light for those below. It is easy to see that this is the least efficient arrangement as the effect of lower leaves is vastly reduced. Instead, we could use 90° as our angle of divergence so that the leaves are more spread out; although this is a better solution it is still far from optimal as there is a complete overlap after the 5th leaf grows. This angle of divergence can be represented as one quarter, such that 1 is equal to a full turn. The denominator in the simplest form shows us how many leaves can absorb light when the source is directly overhead. No matter how small the denominator becomes, overlap is a certainty using these rational numbers. In order to maximise the process, plants have had to use irrational numbers within the angle of divergence as this prevents complete overlap due to their aperiodic nature.

However, not all irrational numbers are equal in regards to their irrationality. Some, like pi for example are well approximated by rational numbers, which in this case would be 22/7. In order to determine the most irrational term we should express the irrational number as a continued fraction. These are sequences in which the a_n values are whole numbers expressed at each stage of calculation.

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

For irrational numbers, these sequences are continuous. Nevertheless, the number can be truncated at any stage to produce a good approximation, provided that the denominator of the next fraction is so large that it is fairly negligible. For some numbers this process would not be appropriate, however, as the denominator may be particularly small. These numbers are deemed to be more irrational due to this property. The most irrational of them being a constant known as phi.

$$\boldsymbol{\Phi} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+.....}}}$$

In order to solve for $\boldsymbol{\Phi}$ we can complete the following steps:

$$\boldsymbol{\Phi} = 1 + \frac{1}{\boldsymbol{\Phi}}$$

$$\boldsymbol{\Phi}^{2} - \boldsymbol{\Phi} - 1 = 0$$

$$\boldsymbol{\Phi} = \frac{1 \pm \sqrt{5}}{2} \approx 1.618$$

The divergence angle within the field of phyllotaxis is known as the golden angle and is equal to approximately 137.5°, the value of 360° over $\boldsymbol{\Phi}^2$. This value can be derived from the Schimper - Braun series, which is formed by the quotient of one fibonacci number and the fibonacci number two places ahead. As the quotient of consecutive fibonacci numbers tends towards $\boldsymbol{\Phi}$ we can calculate the value to which the series converges.

$$\frac{a_n}{a_{n+2}} = \frac{a_n}{a_{n+1}} \cdot \frac{a_{n+1}}{a_{n+2}}$$

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} \cdot \lim_{n \to \infty} \frac{a_{n+1}}{a_{n+2}} = \frac{1}{\boldsymbol{\Phi}^{2}}$$

There is a wide range of other phenomena involving phi within the natural world. Most notably the golden spirals which naturally occur; the nautilus shell and the arms of galaxies both being great examples of this.

Geometry

The golden ratio also plays a prominent role in geometry. Within book VII of *elements of geometry* Euclid writes:

> *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.*

At the most fundamental level, if the greater segment is equal to one then our whole is equal to phi. Obeying the equation:

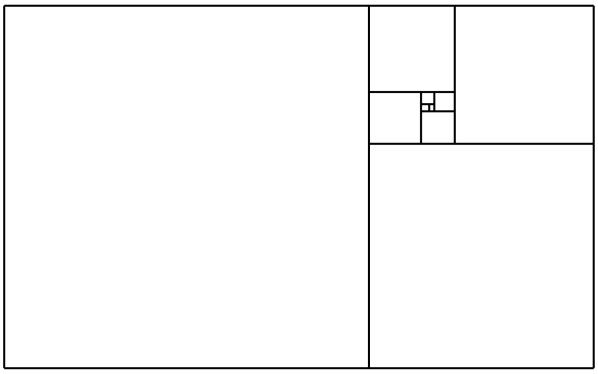
$$\frac{\boldsymbol{\Phi}}{1} = \frac{1}{\boldsymbol{\Phi} - 1}$$

The most prominent use of this value is in constructing the golden rectangle. A rectangle often regarded as that which is most appealing to the eye. It features within a plethora of ancient constructions, such as the *Parthenon of Athens* and *The Gateway of the Sun,* and also within our daily lives as simple items like a credit card.

The golden rectangle undergoes something known as gnomonic growth. As defined by the Hero of Alexandria:

> *A gnomon is any figure which added to an original figure, produces a figure mathematically similar to the original*

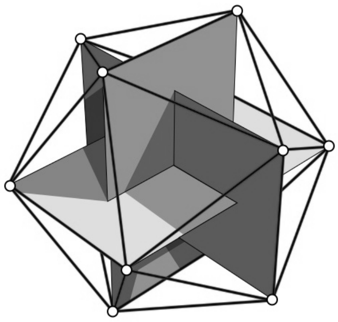
In the case of the golden rectangle the gnomon is a square with lengths equal to its longest side, relating back to the equation seen earlier where the quotient of the sides is equal. Gnomonic reduction also occurs to these same shapes by reversing the process of growth. By removing successive squares from the original gnomon we can produce a shape like that below.



Upon initial inspection, this appears to be remarkably similar to the fibonacci sequence and, in fact, if the fibonacci sequence were to be continued to much greater numbers then this would be its appearance.

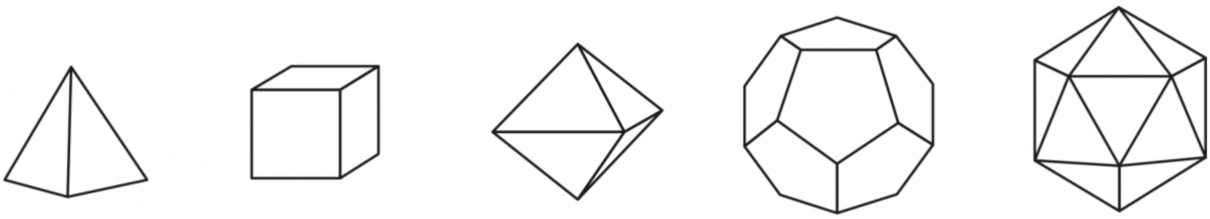
However, there is a more significant phenomenon within this rectangle. If we were to zoom in on the area at which the rectangles become much smaller then we would see that the point of intersection for the diagonals of consecutive rectangles remains exactly the same despite reducing the square by scale factor $\boldsymbol{\Phi}$. All of the golden rectangles converge towards this point of infinite attraction, displaying a property which is exclusive to this fascinating shape.

The golden rectangle can also be used in the construction of a dodecahedron by positioning three of them in perpendicular orientation with their centers intersecting at the middle of the twelve sided shape.

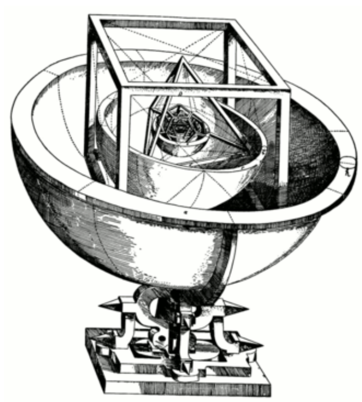


The dodecahedron is one of 5 Platonic solids known to man. These regular polyhedrons are defined as three dimensional shapes with faces that are congruent regular polygons assembled the same way about each vertex. The other four shapes are the tetrahedron, octahedron and icosahedron - each consisting of triangular faces - and the hexahedron, otherwise known as the cube. These shapes can also be inscribed within a sphere with each of the vertices touching the surface.

Originating from ancient Greece, these solids were originally believed to represent the 5 elements which served as building blocks for the universe: fire, water, earth, air, and the cosmos.



In 1597, German astronomer, Johannes Kepler, published the mysterium cosmographicum, a famous book which hypothesised an intrinsic relationship between these shapes and the orbits of the planets about the sun. This was aimed to try and help prove the heliocentric system which was frowned upon during the era due to the religious bias of researchers at the time. The theory is based upon inscribing circles around each of the 5 platonic solids in the following order: octahedron, icosahedron, dodecahedron, tetrahedron, and cube.



These solids can also be paired up with one another to form dual polyhedrons; shapes where the vertices of one correspond to the faces of the other. This property applies to both the cube and octahedron as well as the many sided, dodecahedron and icosahedron.

Both the platonic solids and golden ratio have deep associations within beauty and the natural world, but they are also associated with each other in more ways than one. Phi manifests itself within the volume and surface area of the solids when the edge length is equal to one.

$$\text{Dodecahedral area} = \frac{15\boldsymbol{\Phi}}{\sqrt{3-\boldsymbol{\Phi}}} \approx 20.65$$

$$\text{Dodecahedral volume} = \frac{5\boldsymbol{\Phi}^2}{6-2\boldsymbol{\Phi}} \approx 7.66$$

$$\text{Icosahedral volume} = \frac{5\boldsymbol{\Phi}^2}{6} \approx 2.18$$

It’s fair to say that phi is one of the most incredible constants ever discovered. Not only does it have a prominent feature within plants and platonic solids, but it is also a ratio of pure beauty. These countless occurrences within the natural world are a testament to the extent of its importance in the field of mathematics.

Sources

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