Hidden Sequence

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Have you ever seen a collection of numbers and hoped they were part of a sequence? Have you ever been stumped by a sequence that you cannot find the nth term for? Most people venturing in the path of mathematics, albeit learning it at school or just extra maths for fun, will come across highly complicated sequences or problems that require one to spot a sequence.

This article will discuss a sly trick I have found for generating an unconventional nth term for sequences by taking advantage of the fact that you are given only the first few terms of the sequence.

I will discuss this trick in two cases:
-Combining two separate nth terms of two sequences.
-You can define every terms for the given sequence.

<u>Motivation: Combining two sequences</u>

This case is best demonstrated with an example. Let's say we are given the sequence $u_n = 1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, \ldots$ This sequence is a relatively simple example of this case but is sufficient to demonstrate the trick. From inspection, one case see that this sequence is made of two sequences, one which is defined for the odd numbers and one which is defined for the even numbers. Therefore one can write:

$$u_n = \begin{cases} u_n = (\frac{1}{2})^{n-1} \text{ when n is even and } n\epsilon\mathbb{N} \\ u_n = (2)^{n-1} \text{ when n is odd and } n\epsilon\mathbb{N} \end{cases} \tag{1}$$

This is sufficient to describe the sequence above. However, if one wanted to combine these two sequences into a single equation for the sake of having a sequence that is when the challenge arises. That being said, it is possible to do this. Let the sequence for when n is odd be defined as o_n and the sequence for when n is even be defined as e_n such that

$$o_n = (2)^{n-1} \quad e_n = (\frac{1}{2})^{n-1} \tag{2}$$

To visualise what we need to do, we want to add these two sequences and add an operation such that when n is odd, e_n terms 'disappear'and when n is even,

o_n terms 'disappear'. Therefore, we want to modify both these sequences such that they 'appear'when n is odd or even and 'disappear'when n is even or odd respectively. To clarify the meaning of this, e_n appears for all values of n but we want it to only appear for values of n that are even. Therefore we want to multiply e_n by a sequence that makes the odd terms become 0 or as I phrased it previously - disappear.
We want to change e_n and o_n is such a way that they become:

$$e'_n = 0, e_1, 0, e_2, 0, e_3, 0, e_4, 0, ... \text{ and } o'_n = o_1, 0, o_2, 0, o_3, 0, o_4, 0, o_5, ... \tag{3}$$

Now, remembering how we defined u_n, we can see that:

$$u_n = o'_n + e'_n = o_1, e_1, o_2, e_2, o_3, e_3, o_4, e_4, o_5, ... \tag{4}$$

Now the challenge is to find k_e and k_o such that $e_n \times k_e = e'_n$ and $o_n \times k_o = o'_n$
$\Rightarrow k_e = 0, 1, 0, 1, 0, 1, 0, 1, ...$ and $k_o = 1, 0, 1, 0, 1, 0, 1, 0, ...$

These two functions are periodic functions which oscillate between 0 and 1. One possible way of modelling such functions is using trigonometric functions. What is important to realise is that both these functions take n as the input and so we will need to be careful in choosing the function

To begin with, k_o is at 1 when n is an odd number and it is 0 when n is an even number and never goes negative. Therefore we will need to alter the trigonometric function such that its outputs only become 0 and 1 as n varies across the natural numbers. Without loss of generality, let's choose to apply the sine function to model k_o. At $n = 1$, the function is 1 and sine is 1 when $\theta = \frac{\pi}{2}$. Similarly, at $n = 2$, function is 0 and sine is 0 when $\theta = \pi \equiv 2 \times \frac{\pi}{2}$. This seems to suggest that a possible model for $k_o = \sin(\frac{\pi}{2}n)$. However, we face the issue of negatives as $\frac{3\pi}{2}$ give -1 but instead we want 1. A quick fix for this is to simply take the modulus of the function and as the outputs of $\sin(\frac{\pi}{2}n)$ are $0, 1, -1$ the outputs of $|\sin(\frac{\pi}{2}n)|$ will be $0, 1$, and it will follow the pattern of $k_o = 1, 0, 1, 0, 1, 0, 1, 0, ...$ as we had hoped to model. Similarly, instead of taking the modulus, we could have squared (or raised it to any even power) the expression $\sin(\frac{\pi}{2}n)$ which also would change the outputs from $0, 1, -1$ to $0, 1$, because $(-1)^2 = 1, (0)^2 = 1, (1)^2 = 1$. As an artistic choice, I will continue to use the squaring method although there will be a caveat to it later on.

$$\Rightarrow k_o = \sin^2(\frac{\pi}{2}n) \text{ or } |\sin(\frac{\pi}{2}n)| \quad \Rightarrow o'_n = o_n \times \sin^2(\frac{\pi}{2}n) \tag{5}$$

$$\therefore o'_n = (2^{n-1}) \sin^2(\frac{\pi}{2}n) \tag{6}$$

A similar argument can be made about k_e but the trigonometric has a phase difference of $\frac{\pi}{2}$ which means we could model k_e as a cosine wave with the same input. The same issue of negative outputs, an output of -1 mainly, can be resolved with the same method.

$$\Rightarrow k_e = \cos^2(\frac{\pi}{2}n) \text{ or } |\cos(\frac{\pi}{2}n)| \quad \Rightarrow e'_n = e_n \times \cos^2(\frac{\pi}{2}n) \tag{7}$$

$$\therefore e'_n = ((\frac{1}{2})^{n-1})\cos^2(\frac{\pi}{2}n) \tag{8}$$

Now for the final step of writing a singular nth term to define u_n we simply have to combine e'_n and o'_n $\because$

$$u_n = o'_n + e'_n \tag{9}$$

$$\Rightarrow u_n = (2^{n-1})\sin^2(\frac{\pi}{2}n) + ((\frac{1}{2})^{n-1})\cos^2(\frac{\pi}{2}n) \tag{10}$$

We can see clearly now that the when n is even, the nth term for the odd number disappears from the equation giving us the nth term of the even numbers alone and when n is odd the opposite happens. This is a simple parlour trick to combine two nth terms that define a single sequence where each nth term defines the pattern in the even numbers and the pattern in the odd numbers. Is this useful in anyway? Not at all or I don't think so anyway but this does not take away from the fact that it is a very cool.

However, one must mention the caveats. The example given is a special case for a more general and complicated procedure. If S_e is the set of even numbers and S_o is the set of odd numbers then on can clearly see that $S_e \cap S_o \subseteq \emptyset$ ipso facto (as the set of even numbers is the compliment set to the odd numbers where both sets are subsets of $\mathbb{N}$)and because these two sets fully span the $\mathbb{N}$. So that means there will be no overlap between the two distinct nth terms and there will be no natural numbers missed by using these sets. This example was used to demonstrate the skill and idea that will be used to attempt to generalise this for sets that overlap and incomplete sets.

Periodic values

Another similar sequence to the one discussed previously is a sequence where a single term repeats periodically. Let $u_1, u_2, \gamma, u_4, u_5, \gamma...$ be the sequence in question where the periodic value is γ and u_n is the the other sequence. An important thing to note is that the u_n has discontinuities as the values of n such that the period of $\gamma|n$ will completely disappear from the sequence rather than being shifted along by one - this would mean that we are looking at a sequence such as $2, 4, \gamma, 16...$ rather than $2, 4, \gamma, 8...$ as the latter would require me to define specific shifts for specific values which would require the use of the "big brackets" which would defeat the entire purpose of this article. Therefore, for the sake of making life simpler we will consider a sequence as defined above.

The initial strategy may be to use the same technique as previously by adding sines and cosines and changing the angle such that it matches with the period, becoming almost like a wave function. However, the issue that will arise is when we want to generalise this further which is when we say we have another repeating term α that occurs at a different period. The issue arises because when $n = [\alpha, \gamma]$ as the neither value will "disappear" and the values there will add rather than only one appearing. A possible way of correcting such error is by adding another cosine or sine with an angle and a period that relates to $[\alpha, \gamma]$ such that when $[\alpha, \gamma]|n$ one of the undesired terms disappears but this route makes life so much more difficult when attempting to generalise this idea as it would be highly case dependant which,again, defeats the purpose of this article. This issue arises because we are not using a single set that spans the $\mathbb{N}$.

This is where the trick come to play. We can simply use modular arithmetic as that would provide us with:
1)A set that will fully span $\mathbb{N}$
2)A set that is periodic

This, whilst it may seem to complicate life, in the long run enables us to achieve the ultimate goal of defining a sequence from an arbitrary collection of numbers. Using modular arithmetic in this fashion is highly unorthodox and , if I were to be honest, quite incorrect. We will be using the mod function to generate numbers for us which are defined as the residues of that mod m where $m\epsilon\mathbb{N}$. Now, if we take all that I have said as acceptable steps to take, which for a computer could be defined as correct, then generalising an expression becomes a relatively simple procedure.
We want make a sequences of this form:

$$u_1, u_2, ..., u_{m-1}, \gamma, u_{m+1}, ..., u_{2m-1}, \gamma, ... \tag{11}$$

To be expressed as the sum of two (or more) sequences i.e.

$$0, 0, ..., 0, \gamma, 0, ..., 0, \gamma, ... \tag{12}$$

$$u_1, u_2, ..., u_{m-1}, 0, u_{m+1}, ...u_{2m-1}, 0, ... \tag{13}$$

The period of γ is m and so if we define this sequence with the use of n mod(m) and this will fully span $\mathbb{N}$. We want γ to appear whenever $m|n \Leftrightarrow$ when $n \equiv 0$ mod(m)therefore we should use a variation of cos. However we cannot use simply the cos of the mod as n mod(m)can take the values $0, 1, 2, 3, ..., m-1$ and thus correction terms must be added to collapse the entire sequences to 0 when n mod(m) is not 0. With cos, zero is achieved at multiples of $\frac{\pi}{2}$ and so we can multiply the initial cos that would give us our γ with m-1 more cos with a slightly different input to provide us with the zeros.
Therefore we can begin to formulate the general expression:

$$\gamma_n = \gamma \prod_{k=1}^{k=m-1} cos(\text{n mod(m)} \times \frac{\pi}{2k}) \tag{14}$$

This sequence will repeat γ with a period of m and otherwise would always be zero as there will always be a term $cos(\frac{\pi}{2})$ which would collapse the entire term to zero just as we intended. Therefore we have generated half of what we wanted and what is left is to generate the rest a sequence for u_n. This one follows a similar logic but a different route as we want "disappearance" to occur when the residue is 0 and we want the sequence to "appear" when the residues are $1, 2, 3, ..., m-1$ whilst not affecting the actual value of u_n. One way to approach this is by using sine with the input being only n mod(m)and we divide that by the same sine and input which leaves us with one. Therefore the planned expression thus far is:

$$u'_n = u_n sin((\text{n mod m})) \times (sin((\text{n mod m}))^{-1} \tag{15}$$

Now the issue becomes more clear which is that n mod m can be zero and once it is, the sequence becomes undefined as we would be dividing by zero. A clever way of resolving this issue is shifting every terms back by one, applying the mod(m) and then adding 1. This as previously n mod(m)gave outputs of $0, 1, 2, ..., m-1$ with respect to increasing values of n, but the revised version (n−1) mod(m)+1 would give outputs of $m, 1, 2, 3, ..., m-1$ which will cause no issues as the result will always be defined because when n mod(m) gives 0 the denominator will be a non-zero value and so the division will give zero as we intended. Therefore the revised and correct equation will be:

$$u'_n = u_n sin((\text{n mod m})) \times (sin(((\text{n-1})\text{mod}(\text{m})+1))^{-1}$$

(16)

Or to be written more clearly as a 'fraction'

$$u'_n = \frac{u_n sin((\text{n mod m}))}{sin((\text{n-1})\text{mod}(\text{m})+1)}$$

(17)

Finally, to express the the initial sequence as a single nth terms we simply add $u'_n + \gamma_n$. If the first sequence is S then:

$$S = u'_n + \gamma_n$$

(18)

$$\Rightarrow S = \gamma \prod_{k=1}^{k=m-1} cos(\text{n mod(m)} \times \frac{\pi}{2k}) + \frac{u_n sin((\text{n mod m}))}{sin((\text{n-1})\text{mod}(\text{m})+1)}$$

(19)

This final horrible expression describes the sequence with the period value. It is also important to note that gamma does not need to be a value, it could be a function of n that is will be defined in the when $m|n$ where m is the period of the periodic function or value. The opposite of this comment can be also true; u_n can also be a repeating value or constant. This idea leads directly to the final trick and discovery - The Hidden Sequence.

The Hidden Sequence

This might come as bathetic but we have pretty much already seen the Hidden Sequence by looking at the Periodic Sequences. Now we are discussing how we can represent any arbitrary combination of numbers as a sequence. In order to reveal the Hidden Sequence we must take advantage of the fact that when we are told to "complete a sequence" very few terms are given. The expectation is to spot a pattern but "The Hidden Sequence" forces a pattern on the collection of numbers by saying that the terms are periodic with a period of m where m is the position of the final terms i.e. if given 12 numbers and we were told to find the pattern, we can force the pattern by saying that this sequences is periodic with a period of 12 meaning that these numbers will repeat or functions which generate these numbers will repeat.Whilst this seems absurd, there is nothing incorrect about it as nothing is no true answer to that question. The person asking the question has an expected solution to this but any other pattern that starts the same way is equally as valid as the later terms are not told but even if later terms are told, we can define out period to be greater than before.
Let there be an arbitrary collection of numbers such as S = a,b,c,[more defined terms],α, β, γ... We are told that this is a sequence. If γ is the mth term, then we can define there to be periodicity of m and thus we can define every terms as having a periodicity of m as well but we must shift their own nth terms in accordance to their residue mod(m). Therefore, working backwards we get:

$$\gamma_n = \gamma \prod_{k=1}^{k=m-1} cos(\text{n mod(m)} \times \frac{\pi}{2k}) \tag{20}$$

$$\beta_n = \beta \prod_{k=1}^{k=m-1} cos(\text{(n-1) mod(m)} \times \frac{\pi}{2k}) \tag{21}$$

$$\alpha_n = \alpha \prod_{k=1}^{k=m-1} cos(\text{(n-2) mod(m)} \times \frac{\pi}{2k}) \tag{22}$$

$\vdots$

$$a_n = a \prod_{k=1}^{k=m-1} cos(\text{(n-(m-1)) mod(m)} \times \frac{\pi}{2k}) \tag{23}$$

Then finally we can define S as:

$$S = a_n + b_n + c_n + ... + \alpha_n + \beta_n + \gamma_n \tag{24}$$

Where the ellipses represent the defined sequence of the values in between. With this, we finally found the sequence - The Hidden Sequence.

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