Imaginary Numbers and the Importance of Perspective

Typically, mathematicians (and physicists) have a stereotype for being very literal and matter-of-fact people. Often people say the reason they liked maths in school was ‘because there was a right and wrong answer’, and they liked the objectivity of it- there was a sense of achievement in getting to this right answer. Equally, for many people maths was, or still is, a struggle and something to be dreaded.
Perhaps they never had that moment where algebra 'clicked' and it always remained a mysterious and elusive entity, or perhaps they found the arduous rote-learning method of doing almost the same question again and again utterly boring and lifeless. This is in part due to the general structure of how maths is taught in this country- and whilst of course there are so many heroic teachers doing incredible work, it is not unfair to say that there is a lot of rigidity built into the system.

What I mean is that there is a focus on learning methods and memorising a process built for a very specific type of question, and therefore eschewing what is probably the single most important skill in mathematics: creative thinking. I've often seen people stumble when they come across questions which they are certainly capable of, and shying away from them because they don't look exactly like what they've been taught to solve. This is a huge shame and limits so many people from realising their potential in maths. However the damage does not stop there- the most enjoyable and rewarding part of maths is always finding your own path through a question that stretches and challenges you beyond your comfort zone, and never experiencing that can remove all the enjoyment from maths as a subject. I am convinced that many people who could have been excellent mathematicians did not even take it beyond GCSE because they were limited by this style of learning.

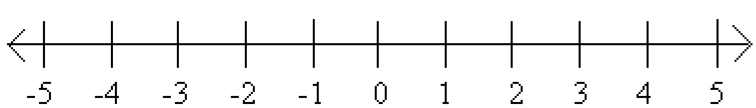
So what is the cure to this? I would say that it is all about a shift in perspective. Maths is at its heart a creative subject and encouraging people to learn the skill of discovering their own methods can develop a much deeper understanding of the subject and foster an enjoyment of maths. And also in a deeper sense, as the title of this essay suggests, it is this ability to think laterally and change perspective that is arguably the most important skill to have in the subject. I am going to use the subject of imaginary numbers to demonstrate some of these ideas.

Why are Imaginary Numbers important?

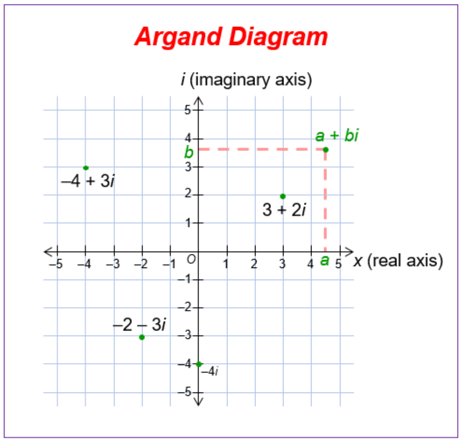
I remember very clearly the first time I ever heard about imaginary numbers. It was, as you might expect, during a secondary school maths lesson where we had been discussing why you cannot take the square root of a negative number. In my teacher’s words, “Mathematicians have invented something called an imaginary number, which engineers sometimes use to make calculations, which does square to make a negative number”. I found this extremely baffling and not at all a satisfactory explanation to say the least. I was in fact quite shocked- I had until that point believed maths to be a proudly objective subject, one where you could quite clearly have a right and a wrong answer, and yet here people were inventing made up numbers and doing calculations with them. What confused me further was his mention of engineers, as I assumed that such an abstract concept could surely only have uses within the esoteric universe of pure mathematics. How could an imaginary quantity have any place in the real world?

Looking back on all this, I do think my maths teacher did a huge (to say the least) injustice to these numbers and their rich history. Firstly, I think the properties that spring out of their use are so varied, surprising and beautiful that they could not have merely been ‘invented’ at all- that is to say, they always existed within mathematics. Secondly, as I just hinted, they are incredibly useful in so many different areas that to confine it to engineering is not at all fair. They turn up in areas as wide range as signal and image processing (through something called Fourier Analysis, also used by Watson and Crick to decode DNA’s structure) and quantum mechanics.

It is easy to see why the education system shies away from discussing these numbers and yet I think that there is no reason why they should be thought of or treated as fundamentally different to, say, negative numbers. In a sense, negative numbers do not really ‘exist’ either, in the sense that we cannot attribute them to a physical quantity such as area or mass. You might consider temperature or debt as an exception to this, however in both those cases the negative sign only assigns a direction to the number, relative to some fixed point. The size or magnitude of the thing they are describing is never negative in and of itself. Similarly, as we will see later on, Imaginary and/or complex numbers can also simply assign a direction to a magnitude, the only difference being the dimensions which they can assign a direction in. So conceptually, they are not as dissimilar from negative numbers as we might be led to think.



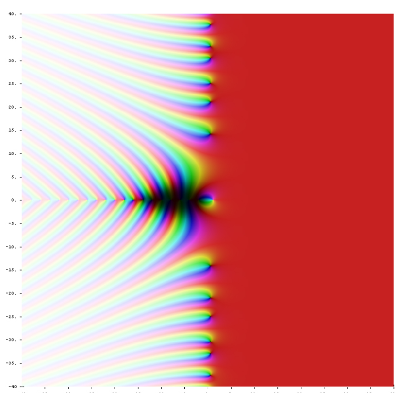
-4 simply means ‘ 4 to the left of zero’. Having a bank statement of -4 pounds (Let’s hope you are not in that position) simply shows you the ‘direction’ the money is to be payed in. The money in there is not actually negative.



3+2i is just like the coordinate (3,2) on a normal graph. We cannot keep adding dimensions to our number line because imaginary numbers complete the whole number system.

The reason I have put a section about complex numbers here is that I believe they are one of the most important examples of how a mathematician can broaden his perspective and think outside the box. A lot of these techniques come under the umbrella of 'Complex Analysis'.

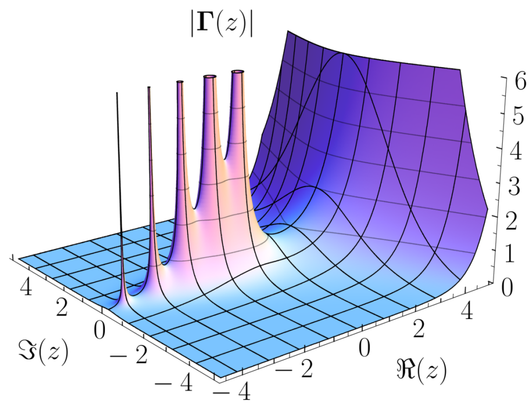
Complex analysis is a truly fascinating area of maths as it can be so counterintuitive but in a really beautiful way. It is mainly concerned with the study of analytic functions. The geometry of these functions is surprisingly rigid, meaning that lots of conditions must be held for a function to be analytic. An example of this is that analytic functions can be thought of as something called a conformal mapping, essentially meaning that any angle between two intersecting lines when transformed remains the same. This however, far from being a limitation, ends up being perhaps the most useful thing about complex analytic functions as it means that information about a small area of the function tells us a lot about the whole entire function.



The famous Reimann Zeta function, the subject of a $1,000000 prize. It begs to be extended into the red area, and due to the rigidity of complex space there turns out to only be one possible way of doing this. This particular analytic continuation gives rise to the astonishing connection between the sum of all integers and -1/12.

This all stems from the idea of differentiability in the complex plane- on a 1D number line we can approach a point from only two directions, making our conditions for differentiability quite simple. However, in a 2D plane we can suddenly approach a point from infinitely many directions. This therefore means that if a function is differentiable in a certain way, or analytic as we said earlier, there are some very strong conditions that must be included in the term.

A particularly creative aspect of Complex Analysis is the idea of contour integration. What is so creative about it is that it utilises this rigid geometry of complex functions in quite astonishing ways that can reveal a lot about real-valued functions. In particular, we can cleverly use something called analytic continuation to extend a real function to the complex plane, and using some of these results we can evaluate otherwise almost impossible integrals without even using integration! The way this works is by examining a certain feature of some of these functions called a ‘pole’: basically a point which blows up to infinity so that it looks a bit like, well, a pole.



A visualisation of a complex function, clearly showing the singularities or ‘poles’ which allow an integral to be computed.

What is so interesting about this, is that one would think that making a two dimensional function into a four dimensional mapping involving imaginary numbers and very strict geometric conditions would just make a problem ridiculously complicated, and yet by taking this strange leap of faith it actually simplifies a lot of difficult problems to something rather manageable. It is, in short, a striking example of how a shift in perspective, although far from being the obvious thing to do, offers a wealth of insight that might have otherwise gone unnoticed. There are almost infinitely many other examples of this sort of idea in maths and I really think that it is so important to realise this and harness its power.

The point I am making is that in maths, answers are often found where we least expect them- and so in a way we should expect that to be the case. There should be a much larger emphasis when teaching maths about being able to see beyond what is right in front of you, and being able to link things together that perhaps are not obviously relevant to each other. The ability to transform a problem into a different mathematical language is an immensely valuable one, and can provide breakthroughs in areas where there might previously have been an insurmountable barrier to progress. I think it is frequently the case that in trying a new approach, something will be learnt from it that can't be learnt from the curriculum. These approaches don't always work, but by attempting them and gaining an understanding of why they do not work, we garner an appreciation of why the other methods do. I really do believe that having the ability to shift perspective is a mathematician's greatest tool- and this idea should be taught from early on, encouraging students to question the methods they are taught and to think for themselves. I think mark schemes and 'model' answers are some of the most prolific prohibitors of creative thought- for there is always another way to do something which could work just as well, if not better. Creativity should be taught as equal in importance to knowing the quadratic formula and rules of surds, as maths is not complete without it.