Mathematical Sequences and Series

Mathematical sequences and series range from simple linear sequences, to far more complex ones, such as geometric sequences. The roles of sequences within society are pivotal, and whilst some are obvious, the impact of mathematical sequences and series can be far more hidden.

Arguably one of the most famous series, Fibonacci's series involves creating each number by adding the two previous numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, etc.

Whilst its roots are mathematical, the series has been used for centuries in many disciplines from music to art. The ratio of subsequent numbers for this sequence is 1.618, which may be called the 'divine proportion' or 'golden ratio'. Furthermore, by drawing squares with lengths equal to each number in the sequence, a spiral – Fibonacci's Spiral - can be formed.

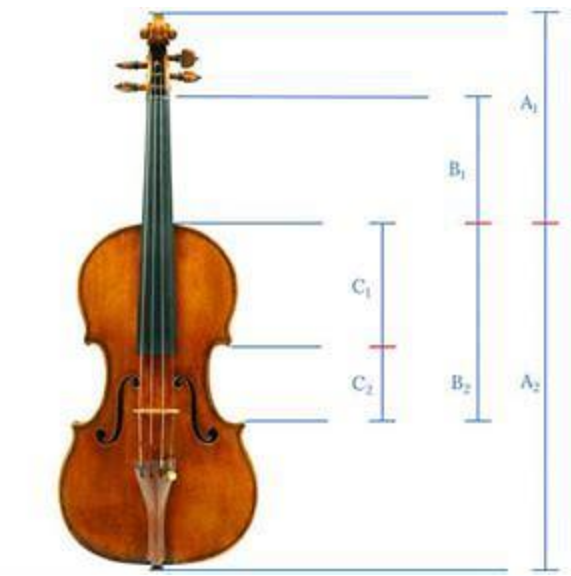
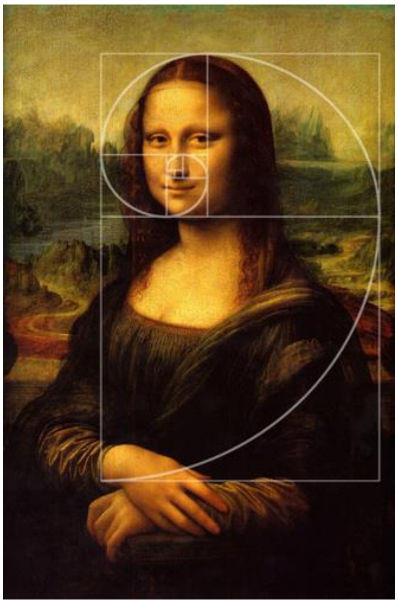


Figure 1

This ratio can be observed in many places within music. For example, Mozart, a prolific composer, equated the ratio of exposition to development and recapitulation of his Piano Sonata No.1 in C major, to this ratio. Another example of the golden ratio being used in music, is in the construction of Stradivarius Violins, now worth millions of pounds. The ratios of the lengths of specific parts of the violins form this ratio, as seen in figure 1. Considering the success of both Mozart and Stradivarius violins, illustrates the importance and prevalence of the Fibonacci sequence and the Golden Ratio it forms, in music.

Fibonacci's sequence and the Golden Ratio has also be utilised in art. Polyclitus' stature of Doryphorus, is a statue that was long considered the standard of harmonious construction and beauty. The proportions of the body of the statue were considered to be the ideal for a human, with the length from neck to waist and knee to foot being equal and the ratio of these lengths to the length from waist to knee forming the golden ratio. This shows how the golden ratio is naturally ingrained in society's standards, and has done so for millennia. Another example of Fibonacci's sequence and the Golden Ratio in art, is the Mona Lisa painted by Leonard Da Vinci. The Spiral is formed by creating rectangles within the corresponding dimensions of 1.618, from each descending Fibonacci Number (8, 5, 3, 2, 1, etc.) and from touching each side in the Perfect Rectangle. Using the rectangle to frame the woman in the painting, the spiral begins at her left wrist, skims over her forehead and continues turning until it reaches the chin, finishing at the tip of the nose, as seen in figure 2. This shows how the use of the Golden ratio within art has



produced one of the most recognisable and renowned paintings. Overall, this shows how Fibonacci's sequence and the Golden Ratio derived from it can be used successfully withing art.

However, the Golden Ratio can also be observed in many places within nature. One example of this, is the ariel view of hurricane Irene in August 2011, although many other hurricanes also display similar patterns. This is because the clouds of the hurricane line up with the Fibonacci Spiral, produced by the sequence, as seen in figure 3. However, this spiral shape can also be observed in living creatures. Although this spiral can be seen on the outside of some animals, such as snail shells or chameleon tails, it can also be seen on the inside. One example of this is in the ovaries of anglerfish, as seen in figure 4. Although, many plants also take this shape, such as the way branches and leaves on tree spiral upwards, or the spiral observed when looking vertically down on pinecones. These indicate that whilst the Fibonacci sequence has applications in music and art, it remains a naturally occurring and fundamental part of our environment.

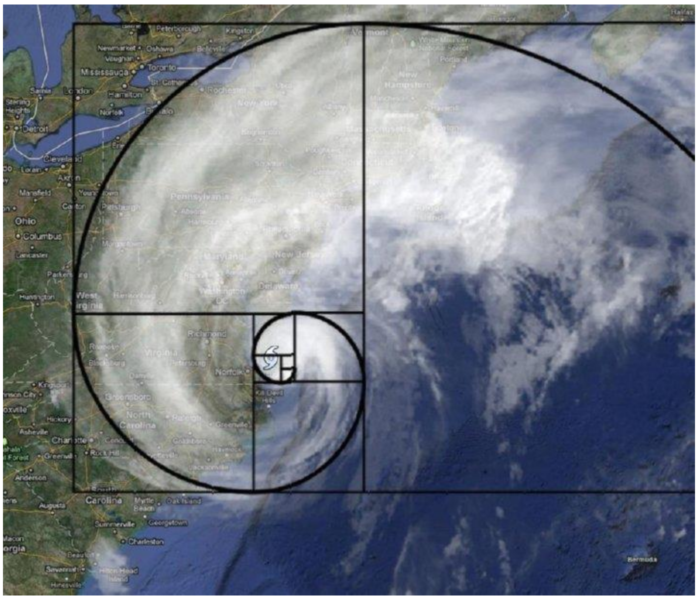


Figure 3

Overall, this shows how although Fibonacci's sequence has simple roots, the uses are varied and endless. From music and art to nature, the Golden Ratio, derived from Fibonacci's sequence can be found in many places, from some of the most mundane objects such as pinecones, to violins worth millions.

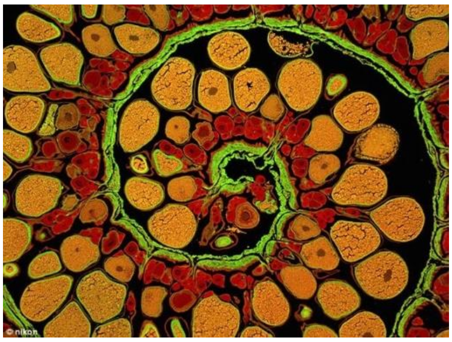


Figure 4

The Fibonacci sequence can be visual and easily understood, appearing frequently within many aspects of society. However, infinite geometric sequences are in contrast to this.

Grandi's sum, the sum of the sequence 1 – 1 + 1 – 1 + 1… , an infinite geometric sequences, can be calculated:

Let $S_1 = 1 - 1 + 1 - 1\ldots$

$S_1 = 1 - (1 - 1 + 1 - 1\ldots)$

$S_1 = 1 - S_1$

$2S_1 = 1$

$S_1 = 0.5$

This means that despite the sequence only containing whole numbers, the sum of the sequence is not a whole number, instead a decimal. Considering geometric sequences to be sequences in the form $a + ar + ar^2 + ar^3$… , the sequence can also be expressed as $1 + (1 \times (- 1)) + (1 \times (- 1)^2) + (1 \times (- 1)^3)$… . Where the value of r is equal to or greater than 1, the sum of geometric sequences can be expressed as 1/(1-r). Using this formula to calculate Grandi's sum, we get:

$S_1 = a / (1 - r)$

$S_1 = 1 / (1 - (- r))$

$S_1 = 0.5$

This sum obtains the same value of the Grandi's sum, using a different method. This validates the value obtained suggesting that the value is valid. Another example of an infinite geometric sequence is the sequence containing all positive integers in increasing order:

Let $S = 1 + 2 + 3 + 4$…

The total of this of sequence can be called Ramanujan's sum, and can be calculated by considering other geometric sequences, including Grandi's sum.

Let $S_2 = 1 - 2 + 3 - 4$…

$S_1 = 1 - 1 + 1 - 1$…

$S_1 - S_2 = 1 - 2 + 3 - 4$ …

Therefore, $S_1 - S_2 = S_2$

$2 S_2 = S_1$

$2 S_2 = 0.5$

$S_2 = 0.25$

$S - S_2 = (1 + 2 + 3 + 4…) - (1 - 2 + 3 - 4…) = 4 + 8 + 12 + 16… = 4 (1 + 2 + 3 + 4…) = 4 (S)$

$S - 0.25 = 4 S$

$3 S = - 0.25$

$S = - 1 / 12$

This shows, that although the sequence contains only positive integers, the sum of the sequence is a negative fraction. This contradicts the idea that a positive integer is obtained from the addition of two different positive integers.

Overall, these infinite geometric sequence and their sum show that despite containing only whole numbers, the sum of a sequence can still be a fraction. Furthermore, even when all the numbers in a sequence are positive and the addition of two positive integers should produce another positive integer, the sum of the sequence can still be negative.

To conclude, whilst some sequences are visual and have many applications within both the natural world and the man-made world, others are theoretical and seem to not conform to simple laws of mathematics and algebra. The contrast between Fibonacci's sequence and infinite geometric sequences highlight the diversity and versatility of mathematical series and sequences and their roles in the wider world.