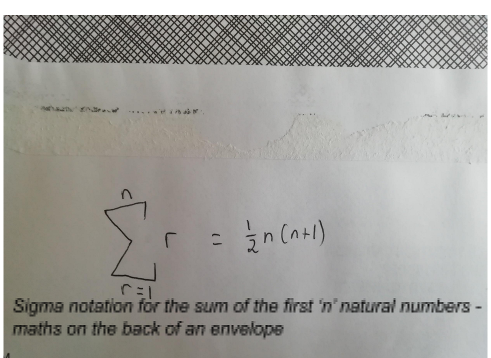
How can we find the sums of simple sequences?

Humans are always looking for ways to be more efficient, and mathematics is often at the heart of this. What would you do if I asked you to do the calculation 1+2? Easy right? What about 1+2+3? Again, you could probably do this in your head even if you're not particularly well versed in mathematics. The same could probably be said for 1+2+3+4. But what if I said add all the integers from 1 to 100? This would take you quite a while, even with a calculator; you would be typing 192 digits, excluding operators. What if I told you I could type 7 digits (with a tiny bit of mental addition) into my calculator and solve this, giving me an answer of 5050? Well, you can just use the formula $\frac{1}{2}n(n+1)$ for any sequence of consecutive natural numbers (positive integers) which starts from 1 and ends at '*n*'. "So what?" I hear you say "That's only for a very specific sequence". Well, we can derive formulae for different sequences - for example the sum of the first '*n*' square numbers - (meaning we start at 1^2 , then 2^2 etc. up to n^2, and add them all up) is $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ which factorises to $\frac{1}{6}n(n+1)(2n+1)$.

So which mathematician spent their precious time finding this out? A primary school age child by the name of Carl Friedrich Gauss (although he was born in 1777 so he is now no longer a school child). The story of how he first applied this is the perfect anecdote for humans trying to be more efficient (and trying to wriggle their way out of trouble). His teacher was attempting to punish him by giving him the exact task I outlined above - the sum of the first 100 natural numbers. I think we can all agree the teacher was probably not best pleased when within a few moments, the young Gauss had come back with the number 5050; solving what should have taken quite a bit longer! Who said discipline doesn't teach you anything? I think this perfectly shows that some mathematics only requires some imagination and clever thinking to make some really interesting connections seeing as a schoolchild came up with such a helpful formula (although, admittedly, Gauss was no ordinary schoolchild).

Before we move on, I must warn you that the next section might be a little bit more complicated; with a fair bit of algebra, but I have included some diagrams to make it a little easier on the eyes, and have tried to explain (simply) what is happening at each step. To help, here's a little overview of what each letter we are using stands for:

r - if we were to substitute a number for *r* for the sequence formula, we would find the value for that position in the sequence (e.g if for the sequence defined as 2*r* we wanted to find the 5th value in the sequence we would set $r = 5$ and find that the 5th value in the sequence is 10)
n - the final position in our sequence (e.g if we were to say find the sum of the first 10 natural numbers, 10 would be our value for *n*)



Sigma notation for the sum of the first 'n' natural numbers - maths on the back of an envelope

So let's have a look at some of the most basic series: standard series. When applied in a mathematical setting, mathematicians use sigma (Σ) notation to denote any sum, but when finding the sum of a series up to '*n*' we denote it using r as the the rth term in the sequence starting at a value which we assign below the sigma (if starting at the first term, we would say $r = 1$ at the base of the sigma). We then put the formula for the value of the rth term to the right of the sigma (i.e for the sum of the first n natural numbers would just be r, and the first n squared numbers would be r^2). Essentially, what we are doing is instead of substituting 1 into our sequence formula, noting down the answer, then doing the same for 2, then 3 and so on up to n before summing them, we are using general formulae which describe this in terms of 'n' (the last position in the sequence which we wish to sum). However there are limitations, since there are not direct formulae for every sequence. We can combine standard series formulae to calculate slightly more complex sequences such as $r^2 + 3r$, which would just be the sum of r^2 added to 3 times the sum of r. These original formulae can be derived or proved to be true from a method known as 'the method of differences'. This aims to turn a sequence's rule into something such as $r(r+1)(r+2)-(r-1)(r)(r+1)$. Essentially, we want part of the formula to be subtracted. With this we can write out what each value of the sequence is worth and we can cancel out certain terms (since we have subtraction). These usually pair up between certain values of r consistently, and once we find which terms cancel with which, we can look at the end of the sequence (considering it as 'n' being the greatest point in the sequence). So to derive the value of the sum of the sequence $r(r+1)$ we can convert it to $\frac{1}{3}((r)(r+1)(r+2)-(r-1)(r)(r+1))$ and then write out the terms of the sequence (for now we can ignore the $\frac{1}{3}$ and just multiply the final formula by $\frac{1}{3}$). This gives us:

When $r = 1$: $(1 \times 2 \times 3) - (0 \times 1 \times 2)$
When $r = 2$: $(2 \times 3 \times 4) - (1 \times 2 \times 3)$
When $r = 3$: $(3 \times 4 \times 5) - (2 \times 3 \times 4)$

From this we can quite quickly spot that the positive part of one term is the same as the negative part of the next - so they will cancel out!

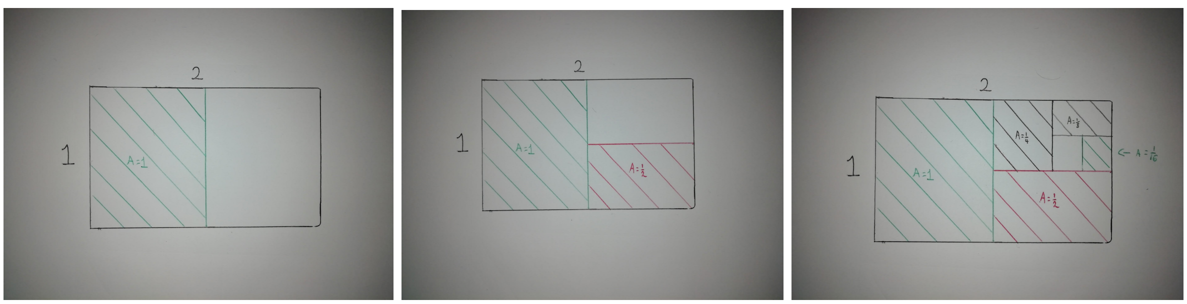
With this information we can look at the end of the sequence:

When $r = (n-2)$: $(n-2)(n-1)(n) - (n-3)(n-2)(n-1)$
When $r = (n-1)$: $(n-1)(n)(n+1) - (n-2)(n-1)(n)$
When $r = n$: $(n)(n+1)(n+2) - (n-1)(n)(n+1)$

Here we can cancel out the negative term for $r = n-2$ with the previous positive term (which we haven't written down, but the pattern dictates that it is the same) and the rest of the terms cancel with one another, except for $(n)(n+1)(n+2)$. This results in only two terms remaining - $n(n+1)(n+2)$ and $(0 \times 1 \times 2)$. Since $0 \times 1 \times 2 = 0$, the formula for this series is $\frac{1}{3}n(n+1)(n+2)$; we must multiply by a third to get our original sequence.

There is more to series than just this however. If we were to look at sequences such as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ …… we would find that this sums to 2. But how? Surely if we are adding an infinite number of numbers we will reach an infinite value - right? The problem here lies in the fact that these fractions decrease too rapidly to reach an infinite sum (we say that the series converges).

Think of it this way, if I have a rectangle, (for ease let's imagine that it has sides 1 and 2 - so it is 2 squares of side length 1 stuck together). This rectangle has an area of 2 (2 x 1 = 2), so let's see what happens when we place quadrilaterals with an area of 1, ½ , ¼ , ⅛ etc. When we add the first quadrilateral, which will be a square, we fill 1 unit of area, and are left with 1 unit of area.Then we add the ½ unit, and ½ is left, then ¼ which leaves us with ¼. Do you see where I'm going with this? If not, have a look at the diagrams below. As you can see, we are always left with the same unit of area that we just added to fill the rectangle, so only when we reach fractions that are negative can we ever exceed 2, but of course, negative area doesn't exist in the real world. Hence, we cannot go past an area of 2. Although of course we can never *actually* fill this rectangle in the real world, in the amazing world of maths we can say that as *n* tends to infinity, we reach 2. The fractions can essentially become 1/infinity, and therefore become infinitesimally small, and indistinguishable from 0 in this problem (even though 1/infinity is undefined).



As you can see, whichever area we add in this progression results in the same area being left over

There are also some intriguing series which result in some famous mathematical constants - such as for π/4:

1 - 1/3 + 1/5 - 1/7 + 1/9 etc. = π/4: π (pi) is extremely important. It is the ratio between the diameter and the circumference of a circle, and is applied in many formulae besides. Without it, the world around you would not be the same; there would certainly be fewer round components to buildings.

Or for *e*, or Euler's number, which has some very amazing properties, such as for the exponential graph y = e^x, the gradient of the line is the line itself. '*e*' can be expressed as the infinite series: $\frac{1}{r!}$

(the little exclamation mark represents a factorial - which means all the numbers from 1 to that number multiplied together, so 4! = 1 x 2 x 3 x 4)

Even if you are not particularly well-versed in mathematics, it is hard to not find it satisfying and intriguing when you can recognise different bits of mathematics coming together.

So let's get back to the most important bit: how can you effectively use this as a party trick to impress people? Obviously the last section was not so essential if all you wish to do is impress your friends with some seemingly genius arithmetical speed. So, if you don't want to try and derive your own formulae, here's a few of the standard series formulae, but use them wisely: we don't want everyone finding our secrets. Why do you think it all looks so complicated?

For the first 'n' natural numbers (sum of r): $\frac{1}{2}n(n+1)$
(Half the final position, multiplied by one more than the final position)

For the first 'n' square numbers (sum of r^2): $\frac{1}{6}n(n+1)(2n+1)$
(One sixth of the final position, multiplied by one more than the final position, multiplied by 2 lots of the final position plus one)

For the first 'n' cubic numbers (sum of r^3): $\frac{1}{4}n^2(n+1)^2$
(The final position squared, multiplied by one more than the final position squared, all divided by four)

Sources:

Fermat's Last Theorem - Simon Singh
The Millennium Problems - Keith Devlin
e (Euler's number) - Numberphile