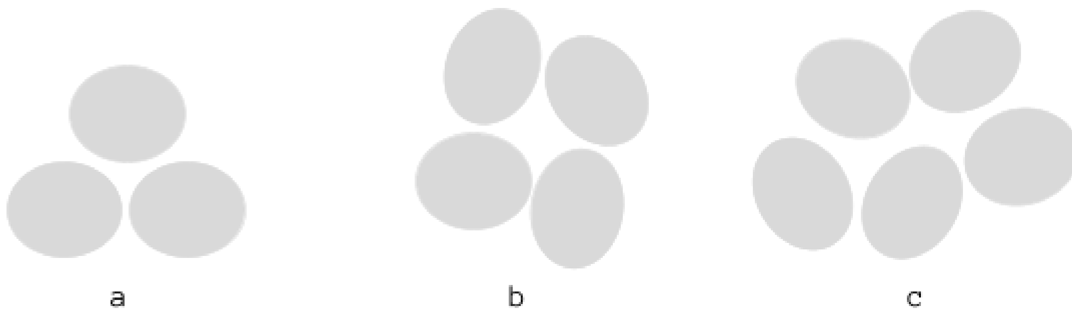
Nim, Binary Numbers and Variations

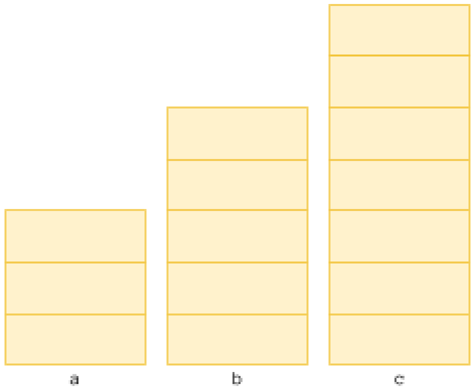
Let's play a game: I've made three piles with 3, 4, and 5 pebbles respectively. We each take alternating turns removing pebbles from the piles. You have to remove at least one pebble each turn, but you can remove as many pebbles as you'd like, as long as they are from the same pile. Whoever removes the last pebble wins.



Games are one of the best ways we can directly interact with math, learning how numbers work and interact with one another. This game was named "nim" by Charles L. Bouton in his 1901 journal article "Nim, A Game with a Complete Mathematical Theory" as part of the *Annals of Mathematics*, in which he explained the winning theory behind the game. Bouton and his paper helped give birth to the field of combinatorial game theory, a branch of mathematics that studies and analyzes strategic sequential two-player games like nim.

Nim is an impartial game, meaning that it is a game where the possible moves are the same for each player in any position. This is different from a partisan game like chess, which is not impartial because players can only move pieces of their own color. Additionally, nim is a game with perfect information, meaning that both players know everything about the current state of the game. These conditions make it so one player is always guaranteed to win so long as they make the optimal plays.

Let's look at an example of a game with three piles:



Starting Position

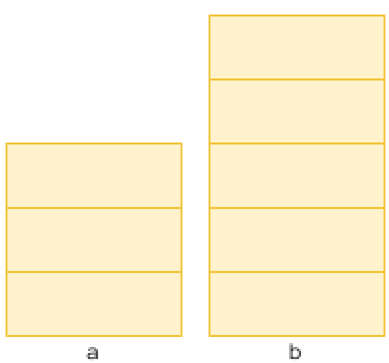
a	b	c	Moves
3	5	7	Player A takes 1 from b
3	4	7	Player B takes 3 from c
3	4	4	Player A takes 2 from a
1	4	4	Player B takes 2 from c
1	4	2	Player A takes 3 from b
1	1	2	Player B takes 1 from a
0	1	2	Player A takes 2 from c
0	1	0	Player B takes the 1 remaining piece in b and wins

Pile sizes / Moves

At first glance, nim seems like a game like chess or reversi that takes "a minute to learn and a lifetime to master," requiring the study of complex theory and accounting for the existence of numerous viable strategies. However, the theory of nim is surprisingly simplistic, having much to do with powers of 2 and binary numbers.

The strategy for one-pile games is intuitive enough. The first player can win by simply taking all of the pieces in the pile. With two piles, the strategy is a bit more complicated. If the game starts with two piles with an unequal number of pieces, then the first player can win by taking the number of pieces from the largest pile that results in the two piles having an equal number of pieces. The second player is forced to unequalize the piles with their turn, as they are only able to take pieces from one of the two equal piles. From there, the first player can equalize the piles again, repeating the process until the first player takes the last piece(s).

This sequence is pictured here:



Starting Position

Pile sizes / Moves

a	b	Moves
3	5	Player A takes 2 from b
3	3	Player B takes 1 from b
3	2	Player A takes 1 from a
2	2	Player B takes 1 from a
1	2	Player A takes 1 from b
1	1	Player B takes 1 from b
1	0	Player A takes the 1 remaining piece from a and wins

In three-pile games, the strategy becomes less intuitive. To determine the optimal moves in games with three piles or more, we need to learn how to use binary numbers.

Binary numbers are numbers expressed in the binary numeral system, also known as the base-2 numeral system, which uses two unique characters "0" and "1" to represent numbers, as opposed to the decimal/arabic numeral system which uses 10 characters. They are often used in computers because of its simplicity—the numbers "0" and "1" commonly used to denote "off" and "on" respectively. All numbers are represented as a sum of powers of two: 2^n (as opposed to the decimal system which represents numbers as the sum of powers of ten: 10^n).

For example, the numbers 0-7 represented in binary are:

Decimal Number	Decimal Expanded ($2^2 + 2^1 + 2^0$)	Binary Number
0	0 + 0 + 0	000
1	0 + 0 + 1	001
2	0 + 2 + 0	010
3	0 + 2 + 1	011
4	4 + 0 + 0	100
5	4 + 0 + 1	101
6	4 + 2 + 0	110
7	4 + 2 + 1	111

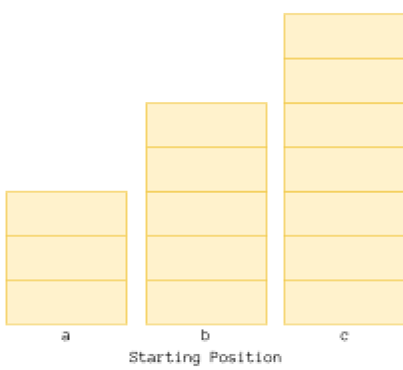
```
pile a    3  => 011
pile b    5  => 101
pile c    7  => 111
-------------------
          1     001

thus, 3 ⊕ 5 ⊕ 7 = 1
```

The general strategy for nim is for the player to manipulate the position into a winning position, with every successive move of the player landing on one of the smaller winning positions. The winning positions for each game are dependent on the number of piles and the number of pieces in each pile. They can be calculated by converting the piece number in each pile to a binary number, then adding the binary numbers digit by digit, without carrying over (binary addition, modulo 2). In combinatorial game theory, the resulting sum is called the nim-sum, notated as $x \oplus y$.

In order to win, the player's move must result in a position with a nim-sum of 0. This is because the winning position is achieved when there are no more pieces left, and the nim-sum of a position with no pieces is 0. Additionally, if a player starts in a position that already has a nim-sum of 0, any moves the player makes will result in the nim-sum being a non-zero value. Thus, if player A makes a move resulting in the game's position to have a nim-sum of 0, the next move by player B cannot leave the game position with a nim-sum of 0. Hence, Player A can always make a move that results in nim-sum 0 if the previous position had a non-zero nim-sum. As a result, eventually player A will be able to take the all remaining piece(s) from the board, leaving the game position with a nim-sum of 0, and win the game.

The player who has the winning strategy (the advantaged player) is determined by the starting position of the game. If the nim-sum is not 0 at the start of the game, the first player has the winning strategy. If the nim-sum at the start of the game is 0, the second player does. Because the advantaged player is guaranteed to win if they play optimally, the other player cannot win unless the advantaged player makes a mistake.

With this in mind, let's take another look at our original example:

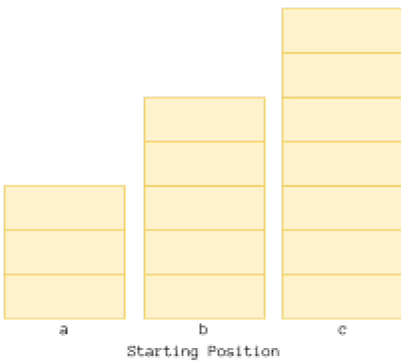


Pile sizes			Nim-sum	Moves
a	b	c		
3	5	7	001	Player A takes 1 from b
3	4	7	000	Player B takes 3 from c
3	4	4	011	Player A takes 3 from a
0	4	4	000	Player B takes 3 from c
0	4	1	101	Player A takes 3 from b
0	1	1	000	Player B takes 1 from b
0	0	1	001	Player A takes the 1 remaining piece in c and wins

Above, we discussed how to win a normal play game of nim. However, nim can also be played as a misère game, in which the player who takes the last piece loses. This change only affects how the last move by the advantaged player is played.

Misère strategy does not deviate from that of normal play until the disadvantaged player leaves only one pile with two or more pieces—all other remaining piles having only one piece. The advantaged player would remove all or all but one piece, depending on the parity of the number of piles left, from the pile that has two or more pieces, so none of the remaining piles would have more than one object. This way, the players are forced to alternate removing the last piece of each pile until there are no more pieces to remove. Whereas, in normal play, the advantaged player removes the appropriate number of pieces that would leave an even number of remaining piles with only one piece, in misère, the advantaged player removes the number of pieces that would leave an odd number of piles with one piece.

Revisiting our example as a misère game:



Pile sizes			Nim-sum	Moves
a	b	c		
3	5	7	001	Player A takes 1 from b
3	4	7	000	Player B takes 3 from c
3	4	4	011	Player A takes 3 from a
0	4	4	000	Player B takes 3 from c
0	4	1	101	Player A takes 4 from b
0	0	1	001	Player B takes the 1 remaining piece from b and loses

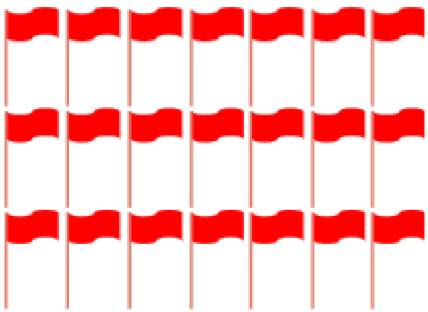
In “greedy nim” players are restricted to choosing pieces only from the largest pile. If there are two or more piles that have the largest number of pieces, then the player can take objects from either one of the piles.

In normal play greedy nim, the advantaged player is the player whose move results in an even number of largest piles. This forces the disadvantaged player to unequalize the piles, allowing the advantaged player to repeat the process until the last piece is taken. This is similar to the two-pile game strategy, in which the advantaged player moves to make the two piles equal to retain the winning position, but with the slight modification that the two equal piles must be the largest ones.

There are also variations of nim that place a limit to the number of pieces that can be removed in one turn, commonly referred to as “bounded nim.” In its simplest form, there is only one pile. Players take turns removing a number of pieces up to a certain upper bound until the last piece is taken (bounded nim can also be played as normal play or misère). The most famous example of this is the game *Thai 21* featured on the American TV show *Survivor*, in which two teams took turns taking one to three flags from 21 total flags until the last flag was taken.

To win this variation, take the remainder of each pile when the number of pieces in each pile is divided by one more than the upper bound (in the *Survivor* example, this would be 3+1, or 4). Then, calculate the nim-sum of the piles using the remainders. From there, the game can be played as it would in the original normal play variation, with the advantaged player making moves that result in the remainders' nim-sum being 0.

The 21 Flags example from *Survivor*:



Starting Position

Pile size a	Remainder divisor (3+1)	Remainder nim-sum	Moves
21	1	001	Player A takes 1
20	0	000	Player B takes 2
18	2	010	Player A takes 2
16	0	000	Player B takes 1, 2, or 3
...	...	...	Player A takes 4-(player B) moves repeat
4	0	000	Player B takes 1
3	3	011	Player A takes the remaining 3 flags and wins

An example with multiple piles:

Starting Position, take up to 5

Pile sizes a	b	c	Remainder divisor (4+1) a	b	c	Remainder nim-sum	Moves
5	6	7	0	1	2	011	Player A takes 2 from a
3	6	7	3	1	2	000	Player B takes 4 from c
3	6	3	3	1	3	001	Player A takes 1 from b
3	5	3	3	0	3	000	Player B takes 2 from a
1	5	3	1	0	3	010	Player A takes 3 from b
1	2	3	1	2	3	000	Player B takes 3 from c
1	2	0	1	2	0	011	Player A takes 1 from b
1	1	0	1	1	0	000	Player B takes 1 from b
1	0	0	1	0	0	001	Player A takes the 1 remaining piece from a

The overall process used to calculate the nim-sum can be used to determine the winning strategy of any impartial game, as well as some partisan games like Domineering. Furthermore, nim and nim-sums have several applications in computer science, being used in cryptography and error-correcting codes called lexicodes. But beyond applications, there is an inherent beauty in the apparent simplicity of nim and the reliability of binary addition to predict the outcomes of games.

Sources:

https://en.wikipedia.org/wiki/Nim
https://www.jstor.org/stable/1967631?seq=1#metadata_info_tab_contents
http://web.mit.edu/sp.268/www/nim.pdf
https://www.math.wisc.edu/wiki/images/Nim_sol.pdf
https://mathworld.wolfram.com/Binary.html
https://www.youtube.com/watch?v=K_MckZc8VvQ
https://en.wikipedia.org/wiki/Subtraction_game
https://www.youtube.com/watch?v=dUXW3Kh_kxo
https://sites.oxy.edu/ron/math/400/09/The%20Game%20Of%20Nim.pptx
https://iq.opengenus.org/game-of-nim/
https://math.rice.edu/~michael/teaching/2012Fall/nim.pdf

Play three-pile nim online: http://www.cut-the-knot.org/nim_st.shtml