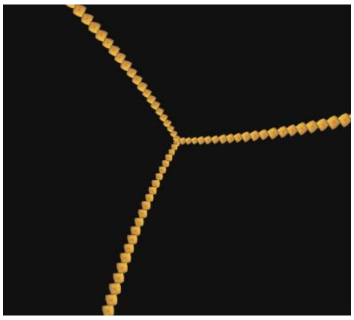
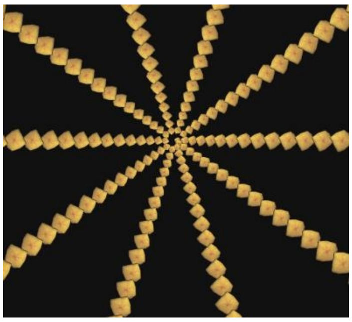
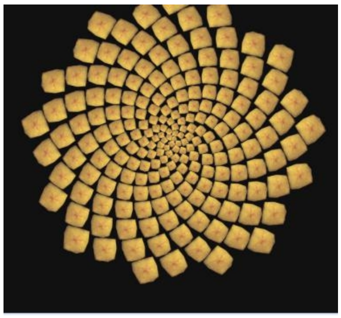
The Divine Beauty of Mathematics: The Golden Ratio

Have you noticed the cover of A-level pure mathematics textbook? Have you thought of why the picture of Mona Lisa is loved by the whole world? Why do the sunflowers locate its seeds in the disc floret in a spiral shape? The answers are all the golden ratio; it exists in the nature, closely related to our life and used in the field of architecture, art and etc.

It is believed that Martin Ohm (1792–1872) was the first person to use the term “golden” to describe the golden ratio in his book “*The Pure Elementary Mathematics*”. Two numbers satisfy the scale of the golden ratio if the ratio of the sum of the numbers, “a” and “b”, divided by the larger number “a” is equal to the ratio of the larger number “a” divided by the smaller number “b”.

To answer the question thrown out initially, how can a sunflower use the limited area in the disc floret, the center of the flower, to maximize the amount of seeds grown? Assuming the center of disc floret is the origin and all the seeds displace around the origin. It is crucial to decide how much of a turn you would have in between each seeds. If you do not turn at all, it is absolutely a waste of space as you will get a straight line. The space is utilized slightly more efficient by rotating 120 degrees between each seeds. Similarly, 1/10 of a turn would provide ten spokes, revealing the number of spoke we get is the denominator of the fraction. Displays with rational number of a turn are under-satisfactory as the space between the spokes are not used efficiently. The ideal pattern would have no gap from start to end.

Irrational numbers break the behavior of spokes. The straight lines are twisted, starting to fill the gaps between the spokes. Which irrational number uses the space most efficiently? The nature answers this mathematical question by using this most efficient allocation on sunflowers. The turn of the golden ratio (0.61803...) fills out mostly the empty spaces available. The fact that golden ratio is the most irrational number makes it special and stands out. In other words, it is almost impossible to approximate it into a fraction.

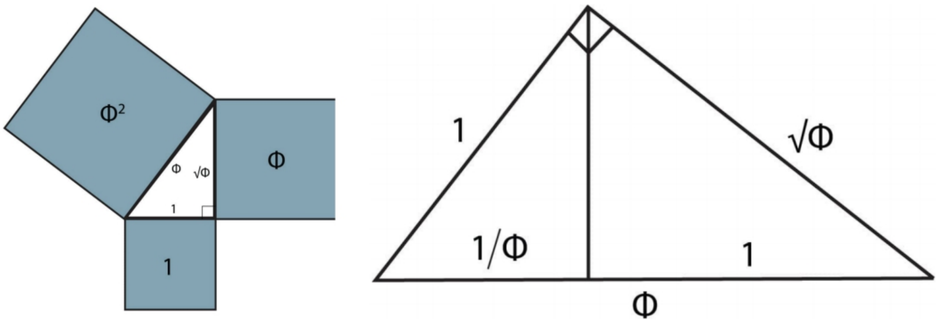
$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}}}}}}}}}$$

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$

The attempt to write irrational numbers into a continued fraction (shown by the picture above) can be used to prove why “phi” is the most irrational number; irrational numbers can all be represented this way with an infinitely long representation, while rational numbers will have finite ones. The continued fraction representation can be calculated by subtracting the greatest integer less than the number and finding the reciprocal of the remainder. Subsequently, this process should be repeated without an end for an irrational number. “pi” is more likely to be approximated than “phi” because of the number 292 at the fifth line of the continued fraction. It can be chopped at the stage of 1/292 as it is a relatively tiny fraction compared with 1/7, 1/15, 1/1 and 1/2 in other stages, showing that the level of accuracy does not change significantly by adding the next level of truncation which is 1/1. Truncating is a method of approximating a decimal number by dropping all decimal places past a certain point without rounding. Therefore, the irrational number “pi” is relatively easy to be approximated at a early stage of 1/292, which makes it not very irrational.

A number that does not have any large number in its continued fraction is more irrational. 1 is the smallest number to purposely keep the continued fraction not ended. If we let this number be Φ , then the number can be further simplified as $\Phi = 1 + 1/\Phi$ since the fraction continues forever. The two roots of $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\Phi = \frac{1-\sqrt{5}}{2} \approx -0.618$ can be obtained by solving a quadratic equation, and are called the golden ratio. To answer the previous question of how to allocate the seeds most efficiently, there should be $\frac{1+\sqrt{5}}{2}$ of a turn between the two seeds.

The quadratic equation $\Phi + 1 = \Phi^2$ is first discovered by German mathematician Johannes Kepler (1571-1630) who was inspired from the Pythagoras theorem. This led to his discovery of a unique triangle, now appropriately known as the Kepler triangle, with sides equal to 1, $\sqrt{\Phi}$, and Φ.

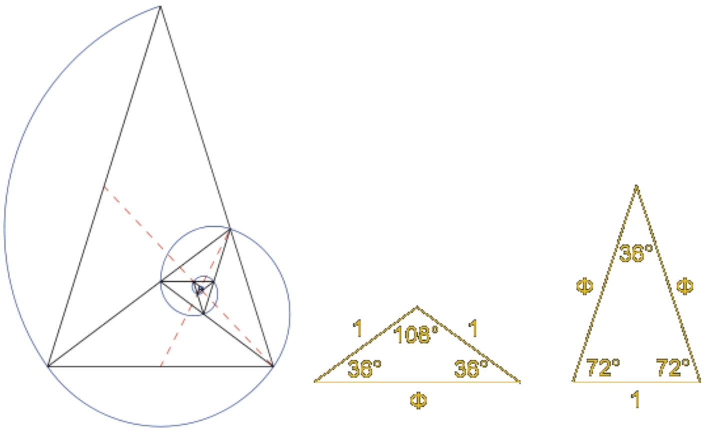


The Kepler Triangle is defined as “If on a line which is divided in extreme and mean ratio one constructs a right-angled triangle, such that the right angle is on the perpendicular put at the section point, then the smaller leg will equal the larger segment of the divided line.” in the book “*A Mathematical History of the Golden Number*” written by Roger Nerz-Fischler. Surprisingly, the Kepler triangle is the only right triangle whose sides are in a geometric progression with a ratio of the square root of the golden ratio, whereas the triangle with side length 3,4 and 5 is the only right triangle whose sides follows a arithmetic sequence.

Other triangles with Golden Ratio proportions can be created with a “phi” to 1 relationship of the base and sides of triangles. The golden triangle, also known as the sublime triangle, is an isosceles triangle such that the ratio of the hypotenuse “a” to base “b” is equal to the golden ratio, $\frac{a}{b} = \Phi$

$= \frac{1+\sqrt{5}}{2}$. Closely related to the golden triangle is the golden gnomon, which is the obtuse isosceles triangle in which the ratio of the length of the hypotenuse to the length of the third side is the reciprocal of the golden ratio. Most interestingly, each golden triangle can be further divided into a golden triangle and a golden gnomon. The hypotenuse of the original triangle is cut into segments "m" and "n", where $\frac{\mathrm{m}}{n}$ = phi, so "m" becomes the hypotenuse of the smaller golden gnomon integrated within the original, and "n" is the base of the new golden triangle. This process can be repeated by cutting the new golden triangle into two.

The golden triangle is used to form a logarithmic spiral. A logarithmic spiral can be drawn through the vertices of the infinite number of golden triangles within the original one. It is constructed by creating an arc that touches the two vertices on the base of the golden gnomon, where the hypotenuse of the golden gnomon is the radius of the arc. This method is repeated in the next smaller golden gnomon triangle, and the process continuous on. In the nature, a logarithmic spiral can be found in snail shells, whirlpools and ferns.



It is known to all that objects that satisfy that golden ratio in their scale are regarded as beauty, but why do they please the eyes? According to Adrian Bejan, professor of mechanical engineering at Duke University, in Durham, North Carolina, the human eye is capable of interpreting an image featuring the golden ratio faster than any other. This kind of image is easily transferable from our eyes to our brain. "This is the best flowing configuration for images from plane to brain" according to Bejan. When it comes to objects that resemble the golden ratio, our brain feels happy and relaxed while we are scanning, recording and interpreting them. When our brain feels satisfied looking at it, those objects are seen as beauty.

It is argued that it is the nature who invented the golden ratio as sunflowers and ferns are good evidence. Mathematicians and architects had been analyzing and using it since the ancient Greek. Nowadays, there is a broad real life application for golden ratio in the field of weapon design, human aesthetics and business, like examining major price movement trends in the stock market.

References

1. R. Herz-Fischler, *A Mathematical History of the Golden Number*, New York, Dover Publications, 1987, p159.
2. G.B. Meisner, *The Golden Ratio: The Divine Beauty of Mathematics*, Race Point Publishing, 2018, p11.
3. C. Budd, “Myths of maths: The golden ratio”, *Plus Magazine* [website], February 23, 2020, <https://plus.maths.org/content/myths-maths-golden-ratio>, accessed March 7, 2021.
4. G.B. Meisner, “The Golden Ratio: Phi, 1.618”, GoldenNumber.net [website], 2020 , <http://www.goldennumber.net/>, accessed March 7, 2021.
5. B. Sparks, “The Golden Ratio (Why is it so irrational)”, *Numberphile* [website], May 8, 2018, <https://www.numberphile.com/videos/the-golden-ratio-why-it-is-so-irrational?rq=golden%20ratio>, accessed 10 March, 2021.
6. “Nature, The Golden Ratio,and Fibonacci too ...”, Math is Fun [website], 2018, <https://www.mathsisfun.com/numbers/nature-golden-ratio-fibonacci.html>, accessed 10 March 2021.
7. K. McVeign, “Why golden ratio pleases the eye: US academic says he knows art secret”, *The Guardian* [website], December 28, 2009, <https://www.theguardian.com/artanddesign/2009/dec/28/golden-ratio-us-academic>, accessed 20 March, 2021
8. Caryl-Sue, “The Golden Ratio”, *The National Geographic* [website], November 20, 2012, <https://www.nationalgeographic.org/media/golden-ratio/>, accessed 20 March 2021.
9. R. Knott, “Fibonacci Numbers and the Golden Section”, *Math.Surrey* [website], March 1996, <http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html>, accessed 20 March 2021.