

The Power of π

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The notion of Pi, or π , can be a somewhat unsettling one; I remember the first time I was introduced to the concept of Pi, it felt like some irregularity with a dissatisfying answer. I recall thinking about how much simpler it would be if it were instead an integer, as I'm sure many student's first thoughts are. But now, having learnt much more about its nature, I've gained a much greater appreciation for Pi and how it mysteriously unifies many branching areas of mathematics under one irrational value. I hope to try and cover some of the history behind Pi which is often left out of the classroom, as well as take a look into what makes Pi so special to us, and the role it plays in places like mechanics and complex numbers. But first, let us start with where Pi came from and how we define it.

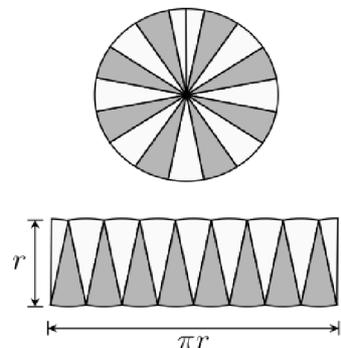
Most students first come across Pi when studying circles, which is how it was first discovered, almost 4,000 years ago. However, the most suitable value for Pi has been a long-lasting debate. Some people propose that the value of Pi should in fact be 6.283... or twice the size of what we call Pi. This constant however has been universally accepted as Tau or τ , which has been popularised by Micheal Hartl's manifesto and is obsessed over due to how it simplifies several mathematical formulae. This is because the value of Tau in radians represents a full 360°, meaning any angle in radians is just that fraction of Tau. Alas, the popularity of Pi being 3.14... has stood the test of time, with its definition simply being the ratio between the diameter and the circumference of a circle, which can be written as:

$$\pi d = 2\pi r = C$$

From this simple definition, we can derive many more equations for Pi. If we also apply a little bit of calculus, we have the formula:

$$\int_0^r 2\pi r \, dr = \pi r^2 = \text{Area}^1$$

These two equations make up the basis of where Pi is derived from, but can be hard to interpret from only algebra. I personally find that the best way to visualise this equation is by splitting a circle up into sectors, and lining them up side by side, alternating them up and down². By doing this, we can see that, as the angle of the segments tends to 0, they more closely form a rectangle of height r and length πr (or $C/2$), which when multiplied, give the desired result, πr^2 . Now with that covered, we're left with the question of how to calculate Pi.



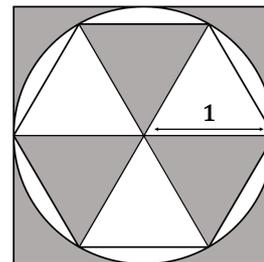
In c.250 BC, a Greek man named Archimedes of Syracuse, famed for his levers, screws and his eureka moment, used polygons to approximate the area of a unit circle³ - a circle with a radius of one and an area of Pi. His approach was to first draw the largest

¹ $[f(x) = \text{Area}] \Rightarrow [f'(x) = \text{Length}]$

² "CircleArea" by Jim.belk - Own work. Licensed under Public Domain via Wikimedia Commons.

³ PBS - Approximating Pi By Rick Groleau

possible hexagon within the circle and calculate its area using the Pythagorean formula (which was only around 150 years old at the time!) Because the hexagon fit entirely in the circle, Archimedes knew that the area of the hexagon would be less than the area of the circle. He applied the same logic to the smallest possible square outside the circle, which he deemed to be just bigger than Pi. Hence Archimedes defined an upper and lower bound for Pi such that $3 < \pi < 4$.



Whilst this may initially sound very inaccurate, the beauty of Archimedes' method is that the more sides the polygon has, the closer the approximation to Pi (since the area of the shape become increasingly circular.) Many mathematicians used this in order to gradually calculate more and more digits of Pi, using polygons with greater numbers of sides. Eventually, this amounted to the work of Ludolph van Ceulen, who published his book 'Van den Circkel' in 1596 which stated an approximation for Pi using a polygon with 4.6 Quintillion sides⁴! This however only yielded a value accurate to 35 decimal places which despite being ample for any real-world applications, seems rather small compared to our present-day calculations, giving Pi to 50 Trillion decimal places⁵. Nevertheless, Archimedes and Van Ceulen's work remains a staple in mathematical history.

Although Archimedes' method lay the foundations for approximating Pi, it was exceptionally inefficient; having to calculate the area of two polygons with several quintillion sides to only generate 35 digits of Pi didn't quite meet the mark for many mathematicians. Globally, some of history's best thinkers took it upon themselves to try and find the best possible approximation of Pi, with contributions from figures such as Ramanujan, Euler, Leibniz and Madhava, all producing work that soon surpassed the Archimedean method⁶. However, my focus lies with Sir Isaac Newton who, at only the age of 23, made significant advancements to western mathematics, such as his work on the binomial theorem which he used to generate a whole new basis for calculating Pi. Whilst he was not the first, he proves a more accessible example of how mathematical methods progressed and brings in ideas that I will extend upon later.

Newton's Binomial theorem is a generalization for the expansion of the sum of two unique terms, raised to some power⁷, such that:

$$(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1} b^1 + \dots = a^n + nab^{n-1} + \frac{n(n-1)a^2 b^{n-2}}{2!} + \dots$$

Initially, it's hard to make any meaningful connection between the binomial theorem, and approximating Pi, but Newton was able to make the logical steps necessary to create a whole new approach. The key into understanding his method lies in the equation for a unit circle, which he rearranged as follows:

$$[x^2 + y^2 = 1] \Rightarrow [y = (1 - x^2)^{1/2}]$$

⁴ [Mathematical Treasure: Van Ceulen on the Circle](#)

⁵ [Calculating Pi: My attempt at breaking the Pi World Record](#)

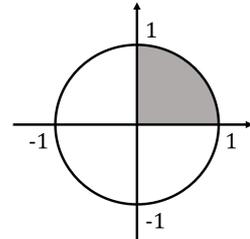
⁶ https://en.wikipedia.org/wiki/Approximations_of_pi

⁷ [Math Is Fun - Binomial Theorem](#)

By presenting it this way, we have an equation for the positive half of the circle with an area of $\pi/2$, which can be applied to the binomial theorem to generate an infinite series:

$$(1 - x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 \dots$$

In my opinion, Newton's next step was equally astute, as he used this knowledge in conjunction with his newly derived fundamental theorem of calculus by integrating this series with respect to 1 and 0. The result of this integral will be a quadrant (shaded grey in the diagram) with an area of $\pi/4$. In this way, Newton generated the following equation:



$$\frac{\pi}{4} = \int_0^1 (1 - x^2)^{1/2} dx = \int_0^1 [1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 \dots] dx$$

And by using the power rule and some cancelling, we can simplify this expression to just:

$$\pi = 4[x - \frac{1}{2} \cdot \frac{x^3}{3} - \frac{1}{8} \cdot \frac{x^5}{5} - \frac{1}{16} \cdot \frac{x^7}{7} - \dots]_0^1 = 4 - \frac{2}{3} - \frac{1}{10} - \frac{1}{28} - \frac{5}{288} - \dots$$

While it might not seem too special at face value, I find this equation to be awe-inspiring! The fact that we are able to express an irrational value as the sum of a series of rational numbers seems unbelievable. What Newton has achieved is proving that Pi can be expressed as a converging infinite series, where each term gets increasingly smaller, meaning it becomes very accurate very quickly. In fact, Newton went on to improve on this formula, by deriving other infinite series with faster convergence to π^8 .

Nowadays, we have a plethora of ways of generating Pi proven by a range of great mathematical minds. Not all evaluations are made equal, and while some series approach Pi rapidly, they tend to do so at the cost of being more complex. Below are a few particularly significant ones - especially the last two, whose variations are most notably used to generate the figure I previously mentioned of 50 trillion digits:

Leibniz Formula⁹:
$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$$

Ramanujan Formula¹⁰:
$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

Chudnovsky Algorithm¹¹:
$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

⁸ [The Discovery That Transformed Pi](#)

⁹ https://en.wikipedia.org/wiki/Leibniz_formula_for_pi

¹⁰ [Ramanujan-Sato series](#)

¹¹ [Chudnovsky algorithm](#)

Today, the algorithms we use to determine Pi are the ones with the most optimal time complexity, though calculating digits of Pi has now become a test of the power of supercomputers, and does little in the way of helping us uncover some of the true mysteries of this number.

Despite Pi's strong connection to infinite series, its tendency to appear in unexpected places can be seen in the Basel problem. The premise was first posed by Pietro Mengoli in 1650, who asked the question, 'what is the value of the sum of the reciprocals of the squares of natural numbers?' This can be written algebraically as:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots = ?$$

This question remained long unanswered for almost a century despite its relatively simple concept. Ultimately, the Basel problem's solution was first presented in 1734 by Leonard Euler, who was notoriously fascinated by puzzles and made large contributions to the ways in which we use Pi today, such as affirming its ties to the field of complex numbers (which I will come on to later). Euler's original method was based on rigorous algebra¹², making the solution rather obtuse, however, more modern approaches make it easier to see how Pi falls into this problem's solution. In particular, my favourite comes from Grant Sanderson - better known as 3Blue1Brown, who contextualizes this infinite series as a set of lighthouses, and uses their light intensity alongside the inverse Pythagorean theorem to compute the solution. He explains his inventive approach excellently in his [video](#)¹³, which I'd highly recommend along with the rest of his channel.

Either way you approach the problem, you end up with a value of $\pi^2/6$, a seemingly unlikely result, and certainly one which wouldn't be expected when first considering the problem. It is fascinating how such a regularly defined series generates the value of Pi in this way, and once again furthers our appreciation for how impressive mathematics really can be. However Euler's obsession with Pi did not stop here; most people know Leonard Euler's name through his work on exponentials and the aptly named Euler's number, whose defining property is that:

$$\frac{d}{dx} e^x = e^x$$

Euler's number shares similarities with Pi, in both its irrational value and ability to be represented as an infinite series¹⁴, and the two are seamlessly combined in what is renowned as the most beautiful and simple piece of mathematics - Euler's equation:

$$e^{i\pi} + 1 = 0$$

The explanation to this equation lies within the complex plane, which is highlighted through the presence of i , which is defined as the square root of -1. i can have a variety of magnitudes, as with any real number, and is represented on an argand diagram, with a real axis and an imaginary axis. This can be a challenging concept to grasp if you've never

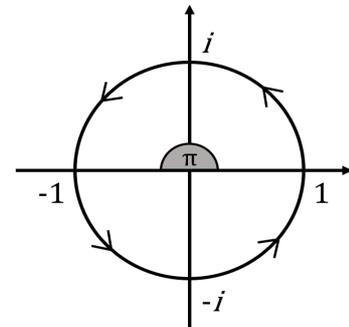
¹² [An infinite series of surprises](#)

¹³ 3B1B's - [Why is pi here? And why is it squared? A geometric answer to the Basel problem](#)

¹⁴ $e = 1/1! + 1/2! + 1/3! + 1/4! \dots$

encountered it before, and for that reason I recommend another of 3B1B's [videos](#)¹⁵ which gives a more in-depth and visual representation of what this equation is truly saying.

By raising something to the power of i , for example, 1, what we are doing is performing a series of multiplications by i , initially setting its trajectory towards i , then continuing to rotate our value on a vector 90° anticlockwise. This transformation takes us around the complex plane in a circular motion about the origin and immediately opens our eyes to why Pi might appear in this scenario. To finish off this explanation, we bring back in Euler's number and its defining properties of differentiation. Here, the derivative is used to describe the rate at which we rotate around the unit circle, and since we are using Euler's number here, much like how the gradient of $f(x) = e^x$ at any point is e^x , we are travelling at a constant velocity. The last piece of the puzzle here is that if we travel a distance of Pi radians around our circle, we finish at a value of -1, and if we add 1 to this we spectacularly land at 0 - our desired result.



In my opinion, Euler's equation is the perfect demonstration of how captivating the beauty of maths is. In one equation we are able to combine 3 of math's most perplexing and irregular constants, as well as the two fundamental values of 1 and 0, into a single line of algebra. Whilst I've tried to give credit to how amazing Pi really is, 2000 words simply don't do it justice, so I've cited some more great explanations of places where Pi really shows its colours. These range from its use in [integrating the normal distribution curve](#)¹⁶, to the [number of collisions between two blocks](#)¹⁷, to how it can be found from simply [dropping needles](#)¹⁸ and many more¹⁹! Although its roots lie within circles, Pi's presence and sense of mystery permeates through the rest of mathematics and is the reason why, on 3/14 every year, I and many others come to celebrate it, on the occasion of Pi day.

¹⁶ Vcubingx's - [Why does pi show up here? | The Gaussian Integral, explained](#)

¹⁷ 3B1B's - [Why do colliding blocks compute pi?](#)

¹⁸ Numberphile's - [Pi and Buffon's Matches - Numberphile](#)

¹⁹ - [Happy Pi Day!](#)

- [Understanding Euler's Formula | BetterExplained](#)

- [The Wallis product for pi, proved geometrically](#)