The Twin Paradox

We say that twins are the same age as each other and have approximately lived for the same amount of time. But what if it was possible that one twin aged faster than the other. Well, it is. This is called the twin paradox. It is a hypothetical situation in which one of two twins travels near the speed of light to a distant star and returns to Earth and discovers that when he returns he is younger than his twin, who remained on Earth. All of this is possible due to the time dilation effect.

In simple terms, time dilation is a phenomenon where time passes slower for an observer who is moving relative to another object or the ‘slowing down’ of the clock of the observer, who is in relative motion to the clock. One may ask how we can show that time dilation is a phenomenon that does truly exists and this is done using maths. In this essay, I will explore how mechanics is involved in the explanation of the twin paradox and time dilation.

Galileo's principle of relativity states:
The laws of mechanics are the same in all uniformly moving reference frames, otherwise known as inertial frames. These are frames that move relative to one another and as we can see each blue axis (x,y and z) is relative to a red axis (x’, y’ and z’).
Figure 1 is an inertial frame, where

$t' = t$

$x' = x - vt$

$y' = y$

$z' = z$

These are all called Galilean transforms.

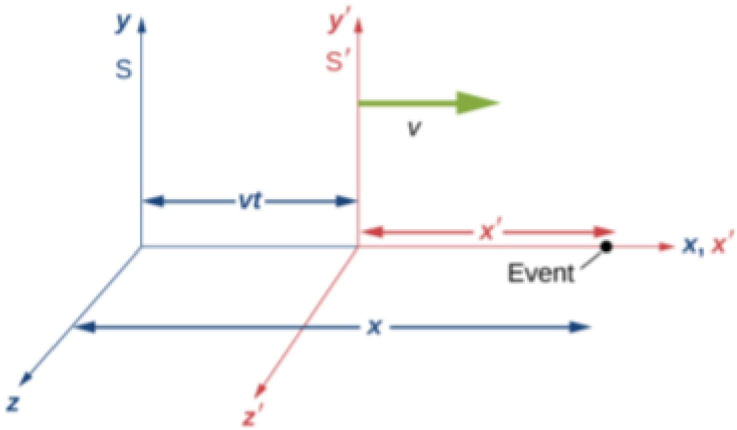


Figure 1

Michelson-Morley Experiment is a failed scientific experiment, where they attempted to detect the velocity of the Earth relative to the luminiferous aether. The aether had been predicted to carry light in a similar way that air ‘carries’ sound.

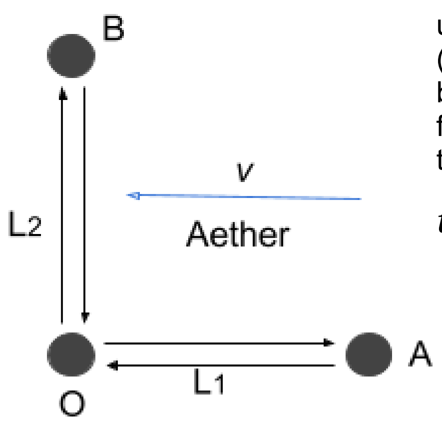


Figure 2

To work out the time of the journey from O to A and back, we use c (the speed of light), v (the speed it is travelling at) and L_1 (distance between O and A). As the direction for O to A and back is parallel to that of the direction of the aether the Earth frame and Aether frame is the same.

$$t_A = \frac{L_1}{c-v} + \frac{L_1}{c+v}$$

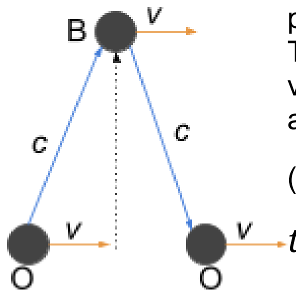
$$= L_1 \frac{(c+v)+(c-v)}{c^2-v^2}$$

$$= \frac{2L_1 c}{c^2-v^2}$$

$$= \left(\frac{2L_1}{c}\right)\frac{1}{1-\frac{v^2}{c^2}}$$

The equation time = speed/distance has been used. The first fraction is the time taken from O to A and the second fraction is time taken from A to O so we add these two together to get the total time of the journey. For the speed of the object (denominators of both fractions), there is +v and -v as the direction changes.

Calculating the time for the journey from O to B and back is not as simple, as the view from the Earth frame and the Aether frame are different. Imagine a boat trying to cross a river horizontally.



Aether Frame

Figure 3

There will always be some movement in the vertical direction from the perspective of the river as that is the direction water in the river is flowing. This is called a 'crosswind'. As the aim of the experiment is to look at velocity relative to the aether, it is appropriate the calculate velocity in the aether frame. Using Pythagoras theorem, we know the 'crosswind' speed (represented by the dotted line) is $\sqrt{c^2 - v^2}$

$$t_B = \frac{2L_2}{\sqrt{c^2 - v^2}}$$

$$= \left(\frac{2L_2}{c}\right)\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Though the difference is usually incredibly small, the time taken for the trip from O to B and back is slightly longer than the time taken from O to A and back.

$$t_A - t_B = \Delta t = \frac{2}{c}\left[\frac{L_1}{1 - \frac{v^2}{c^2}} - \frac{L_2}{\sqrt{1 - (v^2/c^2)}}\right]$$

For this equation to be of any use, the values obtained for L_1 and L_2 would have to be very precise, which is difficult to do.

The spectrometer causing diffraction grating is then turned by 90° so the direction of OA and OB relative to the aether has changed and is now the opposite of each other. If we denote these times as t'_A and t'_B, we will notice something common to the change in the denominators of both times. For t'_A the denominator is that of t_B and for t'_B the denominator is that of t_A. Therefore the calculation for the difference in time taken looks similar to Δt.

$$\Delta t' = \frac{2}{c}\left[\frac{L_1}{\sqrt{1 - (v^2/c^2)}} - \frac{L_2}{1 - \frac{v^2}{c^2}}\right]$$

By finding the difference of the times we no longer have to know L_1 and L_2 to a great degree of accuracy allowing us to use the equation for calculations. This difference is denoted as δ.

$$\delta = \Delta t - \Delta t' = \left(\frac{2}{c}\right)\left[\frac{L_1}{1 - \frac{v^2}{c^2}} - \frac{L_2}{\sqrt{1 - (v^2/c^2)}}\right] - \left(\frac{2}{c}\right)\left[\frac{L_1}{\sqrt{1 - (v^2/c^2)}} - \frac{L_2}{1 - \frac{v^2}{c^2}}\right]$$

As v is much smaller than the speed of light, $v^2 \ll c^2$ meaning it can be approximated using binomial expansion, where $x = \frac{-u^2}{c^2}$ and $n = -1$.

$$\delta \approx \frac{2}{c}(L_1 + L_2)\left[\left(1 + \frac{u^2}{c^2}\right) - \left(1 + 0.5\frac{u^2}{c^2}\right)\right]$$

$$= \left(\frac{L_1 + L_2}{c}\right)\frac{u^2}{c^2}$$

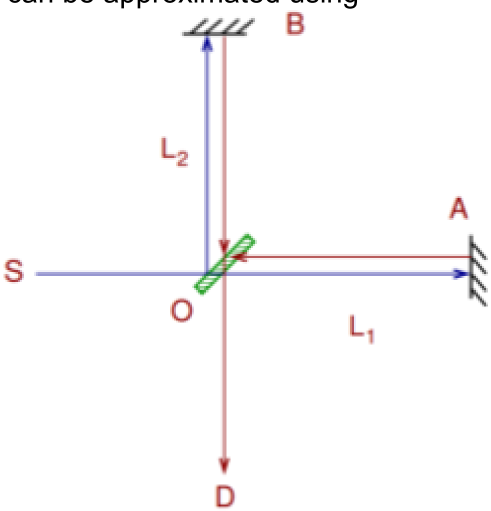


Figure 4

The half-silvered mirror placed at O splits the light and allows it to travel in 2 directions and are then recombined when they return to O and are detected at D. The time difference (δ) is noticed due to a change in position of bright and dark fringes of light caused by interference. The expected shift of fringes is $(c\delta)/\lambda$. For the experiment Michelson and Morley carried out the expected result what a change of 0.4 fringes, however, they saw no change with an uncertainty of ± 0.01, suggesting there is no such thing as an aether and this disproved the hypothesis. From this experiment we have learnt nothing can ‘carry’ light and a few years later Einstein published a paper about this theory of special relativity.

In this paper, which was published in 1905, we are made aware of Einstein’s two postulates:

1. The laws of physics are the same and can be stated in the simplest form in all inertial frames
2. Speed of light, c, is a constant and is the same in all inertial frames

Phenomenons occur at space position and a certain time, introducing us to four-dimensional space-time. These events occur at coordinates in the form of (x, y, z, ct).

Figure 5

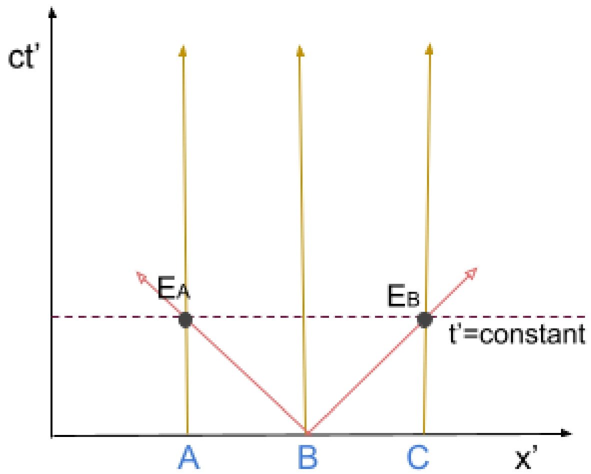


Figure 6

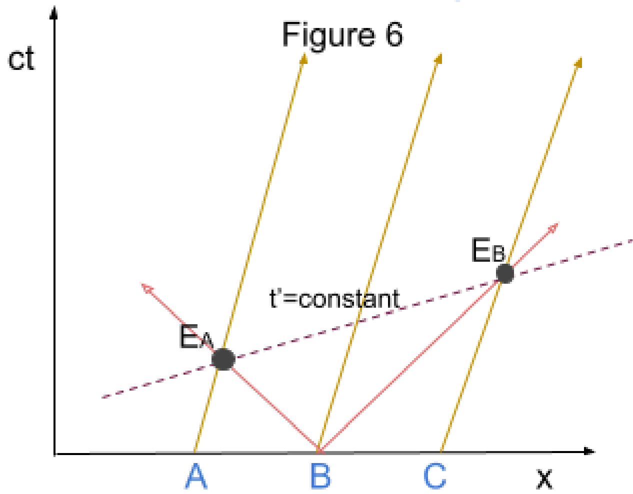


Figure 7

A pulse of light is sent out from B at the same time to A and C, which are the same distance apart from B to synchronise their clocks. The space-time diagram (Figure 6) for inertial frame S’ shows how the time at which event A and event B occurred is the same. The yellow lines represent worldlines and these spacetime curves indicate the position of a particle throughout its’ existence and in this figure specifically A, B and C are stationary therefore their worldlines remain at the same x’ position as time increases. The red light shows the pulse of light travelling away from B and that as time increases, the pulse of light is further away.

If we change the view of the space-time diagram to that of the inertial frame S, worldlines become diagonal as they follow the rule $x = vt + c$, where c is the initial position and this would be their corresponding x’. Due to the second postulate, the red lines remain at 45°from the x-axis as the speed of light is constant. That means in frame S, the pulse of light reaches A and B at different times hence $t' \neq t$. We can no longer use the original Galilean transforms mentioned at the start so these are adjusted.

$x \to x'$ transform is now $x' = \gamma(x - vt)$

γis a constant that allows the transform to remain linear and guarantees $x = vt$. This is needed so $x - vt = 0$ resulting in $x' = 0$in the frame S.

$x' \to x$ transform is now $x = \gamma(x' - vt')$

$x = \gamma(\gamma(x - vt) - vt')$

After expanding the brackets and making t' the subject, the equation becomes:

$$t' = \gamma(t + \frac{1-\gamma^2}{\gamma^2 v}x)$$

γis known as the Lorenz factor and depends on the speed of the object, everyday speeds tend to be close to 1. As $\gamma \to \infty, v \to c$. The closer v is to the speed of light, the greater the Lorenz factor is.

This difference in the elapsed time of t and t’ is time dilation and it implies that as speed is closer to c, the clock becomes ‘slower’ as this time difference of events occurring becomes larger. Using this concept, we can conclude the twin paradox is indeed possible as the clock for the twin travelling near the speed of light is slower than the twin on Earth.