Utopia^{12}

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Just imagine. Children enjoying school. Adults understanding ratios, even better: people understanding statistics and percentages! Just imagine. Just imagine all the possibilities. A child comes home from school and tells their parents about them being in the top third of their class. In one world, a parent might struggle with an infinite amount of decimals; fractions may be an odd thing. But still, many adults rather look at decimals, just like they see them on the news, on bills, and when talking about their new born's age. So a conversion from $\frac{1}{3}$ to 0.333... has to be done. But that darn list of threes doesn't seem to come to an end! And that's not even considering those horrifying percentages!

Just imagine another world, maybe people there have six fingers per hand. Maybe in that world, dividing a whole into three equal parts leaves you with sets of four. In that world, a true utopia, one would understand that fact simply by looking at one's hands. Phrases along the lines of "A third? Oh, that's easy! That's 40%, or 0.4, if you will." would echo far and wide.

Just imagine, how parents wouldn't break out in sweat at the mere thought of having to deal with gruesome fractions and decimals, like $\frac{1}{6}$. In the utopia of the twelve-fingered people, that's just 0.2, easy as π, which, by the way, would look quite a bit different in that world, just imagine!

Just imagine, going to the bank, asking for a loan, and understanding interest. But how, you ask? Not waging war against common fractions like $\frac{1}{2}, \frac{3}{4}, \frac{1}{3}, \frac{5}{6}$ helps a lot to get acquainted with the other fractions. And before long, you have no disadvantage in a bank clerk's office! And all you might need are two extra fingers; two more digits, quite literally.

In our utopia, let us consider the digits $0,1,2,3,4,5,6,7,8,9,\flat,\natural$. There are two odd ones, you say? Well, in the utopia, they are rather natural. You, dear reader, might know them under different names:

$$9+1=\flat$$
$$9+2=\natural$$

Let us consider the age of one of your great-great grandparents, an elderly person of mere 111 years. A bit young for a great-great grandparent, right? After all, I was referring to their age in the world of twelve-fingered people. In the more commonly known world of decimal numbers, this equates

$$1\cdot 12^2+1\cdot 12^1+1\cdot 12^0=144+12+1=157$$

years of age.
Besides reducing a person to their age, let us generalize this concept, and look at numbers of the form

$$(a_n a_{n-1} ... a_0)_{(12)} = a_n\cdot 12^n + a_{n-1}\cdot 12^{n-1} + ... + a_1\cdot 12 + a_0\cdot 12^0$$

with coefficients $a_0, ..., a_n$ taken from the list $0,1,2,3,4,5,6,7,8,9,\flat,\natural$. We shall differ between decimal numbers and these "duodecimals" with a subscript, like the decimal number $157_{(10)} = 111_{(12)}$.
Take a look around, dear reader. There might be a clock somewhere, or a calender. There, you have $20_{(12)}$ hours in a day, $10_{(12)}$ months in a year. Is that a coincidence or does this weird utopia shimmer through here and there? Just imagine!
It is all too understandable that it does seem more than just a bit odd to have the symbols "1-0-0" not represent one hundred of something. In our reality, a tight grasp of the concept of "one-hundredness" is widely spread, especially among people who use the metric system. The idea of "one hundred" seems to be deeply connected to a one followed by two zeros. To even think of this sequence of digits to represent anything else might be, at least for many people, and for lack of a better term, out of this world! In that other world, though, the same can be said of the understanding of the same sequence of digits in base twelve. Just imagine!
Part of the practicability of this number system lies in the number of divisors of the number twelve, which, by the way, even has a name that accentuates its special role. A dozen is divisible by $1,12,2,6,3,4$. We can thus divide our

days, our years, our everything nicely into smaller parts. Like fiscal quarters, each a period of three months.
Tell me, dear reader, what is $0.23_{(10)}$? And no, I am not kidding. Don't just think of this number as a sequence of symbols, try to see beyond that.
Let us take that decimal number apart. 0.2, that is $\frac{2}{10} = 2 \cdot 10^{-1}$, right? So then 0.03, that should be $\frac{3}{100} = \frac{3}{10^2} = 3 \cdot 10^{-2}$. So you see: $0.23_{(10)} = 2 \cdot 10^{-1} + 3 \cdot 10^{-2}$. And we can go on like this forever! Or we could, if we didn't have better things to do.
Better things, you ask? Just imagine! The square root of two. You know... that number which, when multiplied by itself, gives you two. In decimal, it's something along the lines of 1.4142... Just tilt your phone and use its calculator. What would this number look like in our utopia?
$(1 + 5 \cdot 12^{-1})^2$ is a bit too much. $(1 + 4 \cdot 12^{-1})^2$ is about 1.77.., much better! Now, $(1 + 4 \cdot 12^{-1} + 1 \cdot 12^{-2})^2$ is less than 2, but we can do better. If we try $9 \cdot 12^{-2}$ instead of $1 \cdot 12^{-2}$, that's better, but still not quite.
You'll see, dear reader, that we will have to use one of those odd looking new digits. In fact, the duodecimal square root of two starts out like $1.4\natural 7917$....
You figure out the rest.
Speaking of roots of two, not only the twelve-fingered people have an odd interest in its 12th root. Dividing the frequencies of two semitones, one gets $\sqrt[12]{2}$ in a perfect octave, for it has - just imagine - 12 semitones. The whole octave has a frequency ration of 2 : 1, which forces the frequency ratio between two semitones in a perfect octave to be $\sqrt[12]{2}$. But what is it about the number 12 that makes it deserving of adjectives such as "perfect"? It can't possibly be all about the number of divisors, can it?
It seems to pop up in some unexpected places. Besides the ones listed above, its additive inverse occurs as the multiplicative inverse of $\zeta(-1) = -\frac{1}{12}$ for the analytic continuation of the ζ-function. Then there is the number of jury members in trials (should you ever want to fight someone on the importance of 12), or the number of astrological sign. It might even be creeping up right behind you, reader! Look out!
One might argue that some of these findings of the number twelve (except for the one concerning the ζ-function, of course) are of no (mathematical) significance. Many of its practical properties in everyday life are without a doubt owed to 12 being a *highly composite number* - it has more divisors than the numbers 1 through 11. Besides the computational implications this has when it comes to duodecimal numbers, the utopia of twelve-fingered people might not differ too much from our world. Just imagine not having to imagine too

many differences. If the reader still wants to use the duodecimal system, a word of advice: as long as the reign of 10 is not over, one should not try to get used to base twelve too much. Just imagine getting a bill of 100 units in the reader's currency and mistaking it for $100_{(12)}$. That's a difference of 44 units in your currency.
Thus far, aficionados of the duodecimals should probably mostly enjoy them for fun (or at least be careful with the conversion!). After all, ten-fingered people can count using their twelve phalanges per hand, after excluding the thumb, of course. Just imagine!