ABSURDLY LARGE NUMBERS AND OUR FASCINATION WITH THEM

The definition of a number is an arithmetical value expressed by a word, symbol or figure, representing a particular quantity and used in counting or making calculations. They have existed for thousands of years since the stone age era as tally marks. As society progressed and large communities formed larger numbers were required. Eventually we used these numbers to form the basics of mathematics and they were used to predict the size and movements of objects and bodies that are far bigger than ourselves like stars and planets. Nowadays extraordinarily large numbers are used in things like computers and data that is transferred across them. However there are some numbers that are so absurdly large, indescribably so, to the point where they are impossible to even conceive of and make the universe itself feel insignificant.

Most people have heard of googol. It is a 1 followed by a hundred zeroes and was invented in 1938 by 9 year old Milton Sirotta who was the nephew of mathematician Edward Kasner. Although this may not feel like an especially large number it already annihilates the scale of most things in the universe. In fact the predicted number of atoms in the universe is predicted to be around 10 to the power of 80. Googol is 20 orders of magnitude greater than this number. Yet this is the smallest of the numbers that the human mind has invented.

Another famous but less well known number is the googolplex. The concept behind this is simple; it is 1 followed by a googol number of zeroes. This is a number so large that it would take 1.51x10^92 years to write all of its digits by hand, which is 1.1x10^82 times greater than the age of the universe. For some time this was the way to make bigger numbers. Simply adding more zeroes to the end of numbers. However in 1971, a mathematician called Ronald Graham invented something known as Graham's number. This was a number that couldn't even be represented with the power towers that were used to show other very large numbers. This is how it worked.

Our starting number can be 3. If we add 3 to 3 we get 6. If we do 3 times 3 we get 9. 3 to the power of 3 would be 27. Usually this would be represented by 3^3. But for the purposes of this thought experiment we can use arrow notation where it can be written as 3↑3. What happens if we use two arrows? Well 3↑↑3 would be the same as 3↑(3↑3), or otherwise written as 3^{27} which is about 7.6 trillion. If we tried to do 3↑↑↑3 it would be like writing 3↑↑3↑↑3. This might not look big but in actuality it is a 3 to the power of 3, to the power of 3 again and again 7.6 trillion times. This number is already huge and is absolutely unfathomable in and of itself, so it may come as a surprise to think that in comparison to Graham's number it is pretty insignificant; next to zero. To get to the next part of finding Graham's number let's define g1 as 3↑↑↑↑3. This is 3 to the power of whatever 3 to the power of 7.6 trillion threes is. Now let's say that g2 is 3 with a g1 number of arrows. Hopefully you can start to see how you can start to see how this is about to spiral out of control. If this keeps going with each g value with the previous g value number of arrows we can keep going until g64 which is Graham's number. This number is so big that simply attempting to comprehend and process it in your mind perhaps by saying it or writing it all down would cause so much information to be held in your brain that it would collapse into a black hole. All that we know about this number is that its last digit is 7. That is not exaggeration; it is simply the terrifying large scale that some bored mathematicians decided to come up with. Of course we could keep going with more g values but there is a reason why g64 is special. It is because it is an upper bound solution to a problem involving higher dimensional cubes. The problem is as follows: if you connect every pair of vertices a n-dimensional hypercube to create a network otherwise known as a graph and then you colour the edges of the graph red or blue, what would be the smallest number of n for which each pairing has the same colour and each vertices is in an even number of pairings. They worked out that the maximum number people to be Graham's number with the lower bound being six meaning that the solution is somewhere between these.

Understandably, this is quite a lot. Imagine if I said that there is a number that makes Graham's number feel like zero. Feels like a joke right? Unfortunately, I don't make jokes, so moving on to the second exhibit of the day we have TREE(3). This is a number derived from the TREE function and here's how it works. If we have three different types of nodes and a "forest" is made from these with lines as trees. The first tree can't have more than one seed, the second can't have more than two and so on. If you build a tree that has an earlier tree contained within it, it does not count. With one type of seed you can only have one tree, so TREE(1) equals 1. TREE(2) would have 3 kinds of trees. With TREE(3) the number of trees you could have is so unimaginably large that even if you had a Graham's number of people and asked them to imagine an equal part of TREE(3) all of their brains would collapse into a black hole. This is used in Kruskal's tree theorem. Imagine that all the sets of all the different combinations of seeds and if there is an order to it then the corresponding trees built would also have an order. You can't prove it with finite numbers but you can use the TREE function to prove it. To prove TREE(3) is finite you would need 2↑↑1000 symbols to represent it using finite arithmetic. We can't exactly write the whole proof down because the universe would likely end before finishing, but it is theoretically possible, which would not be possible with finite arithmetic alone.

The final number I want to talk about is Rayo's number which is the biggest number that I have recounted so far. Perhaps what makes this even more interesting than just a large number is the fact that it resulted from "Big Number Duel" which was an MIT contest in 2007 between Agustin Rayo and Adam Elga. The idea was who could think of the biggest finite number. The rules were that one person would write a number and the other would have to write a number but each time it would have to be creative rather than just adding 1 each time. It also could not have something like "a really big number"; it had to be purely mathematical. Rayo begins by writing around 30 ones on a blackboard. Elga came to the board and ignored the first two numbers and then rubbed his finger just above the bottom turning it into 11!!!!!!!!!!!!!!!!!!!!!!!!!!!!!, meaning that each factorial would be factorial-ed again and so on. Rayo then wrote down the Busy Beaver function of googol. The Busy Beaver problem that states the maximum number of ones that are printed on tape when it halts using only a given set of states. As an analogy, let's say we have a hotel with all lights off and a robot goes around to switch lights. Let's say that there are two states, the first being a scared state causing the robot to turn on a light in a dark room. If in the scared state he goes into a lit room he transfers to the normal state where it turns the light off. If in normal state it goes into a dark room it shifts to a scared state. However this will end up in a loop and we need something that terminates eventually. So we add a state where it stops turning lights on and off. If the robot has a n number of states the busy beaver number would be the maximum number of lights that are on when the robot stops. For context BB(6) is 3.514 x 10^{18267}. And Rayo had just written down the BB of googol. Meaning our robot had a googol number of states with a different action for each one. This number is even bigger than TREE(googol. To solve this Elga came up with a super Turing machine that could compute the value of BB(googol) meaning it could instantly know the value of the Busy Beaver function. This resulted in Super Busy Beavers resulting from the fact that our new fantasy computer could compute much larger numbers. To handle these a super duper Turing machine was imagined to generate even larger numbers. Rayo then revealed the winning number. Consider any finite number that can be expressed with first order set theory with googol symbols or less. First order set theory is just the language of mathematics like parentheses, dots, variables amongst others. First order refers to singular numbers that can only have one state. One using first order set theory would be $\exists x_1 \forall x_2 (x_2 \in x_1 \leftrightarrow (\neg \exists x_3 (x_3 \in x_2) \vee \forall x_3 (x_3 \in x_2 \leftrightarrow \neg \exists x_4 (x_4 \in x_3))))$. It is much more efficient to describe large numbers and functions. TREE(3) could be feasibly represented with this sort of language. Rayo described a number that used a googol number of these symbols. Rayo's number was bigger than this using second order language in mathematical format:

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∀R {
  {
   ∀[ψ], s: R([ψ],t) ↔ ([ψ] = "xi ∈ xj" ∧ t(x1) ∈ t(xj))
              ∨ ([ψ] = "xi = xj" ∧ t(x1) = t(xj))
              ∨ ([ψ] = "(¬θ)"   ∧ ¬R([θ], t))
              ∨ ([ψ] = "(θ∧ξ)" ∧ R([θ], t) ∧ R([ξ], t))
              ∨ ([ψ] = "∃xi(θ)"  ∧ ∃t′: R([θ], t′))
              (where t′ is a copy of t with xi changed)
  } ⇒ R([ϕ],s)
}
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Whilst there are other numbers that could be larger often they are just adding onto this Rayo's number is perhaps the largest original number that as far as I can tell is reliably the largest finite number ever defined. How long would it take to write it with around googol symbols of set theory? Well if someone wrote each symbol in a Planck time, which is the smallest unit of time in the universe which is around 10^{-44}seconds, (and assuming we already know all the numbers) it would take 10^{56} seconds or 10^{48} years. Whilst this is still a large number it is less than the 10^{80} atoms that we talked about at the beginning. In some ways it is in fact a very small amount of time.

These numbers are huge and will never be practically used at any point in our civilization. So why do we do it? Why do mathematicians spend so much time and effort to create these monstrous creations that ultimately serve no real purpose? This might be the only thing that I can not give an answer for. Perhaps it gives us a sense of power that we can manipulate numbers that oustrip the very nature of the universe. Or perhaps it simply is part of the many weird and wonderful things that mathematics brings with it. Whatever it is, what can be said is that it shows how far we have come from counting the number of cattle with tally marks into the great leaps and bounds that we have made in our society today and that is something that I believe to be worth remembering.