The Hyperbolic Universe

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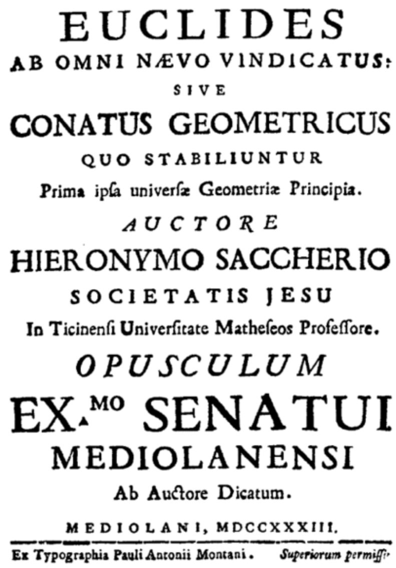
The Jesuit and his Rectangles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CONATUS GEOMETRICUS
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Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permiss

In 1733, Italian mathematician and Jesuit Giovanni Girolamo Saccheri wrote a book, *Euclid Freed of Every Flaw*. He grappled with a simple question: *do rectangles exist?*

To draw a *Saccheri quadrilateral*, start with a base and draw two equal-length line segments going up from the endpoints meeting the base at right angles. Close the figure at the top. *Are the angles at the top also right angles*: is it a rectangle?

Saccheri successfully refuted the possibility that they could be obtuse. Then he attacked the acute case to no avail.

Thirty-two propositions later, he thought he'd cracked it. He wrote "The hypothesis of the acute angle is absolutely false, being repugnant to the nature of the straight line." Alas, his proof doesn't work!

As he felt it necessary to write a Volume 2, he probably realised that at some point. But he doesn't correctly disprove it there either!

Here Saccheri's part comes to an end: he died not long after publishing his work.

Legendre had similarly proven from some axioms that the angles of a triangle could not add up to *more* than 180°; but he could not exclude them adding up to *less*. Lambert had considered his own quadrilateral – a Saccheri quadrilateral divided in half!

No one could disprove the hypothesis of the acute angle, but some persevered and derived some very strange consequences from it:

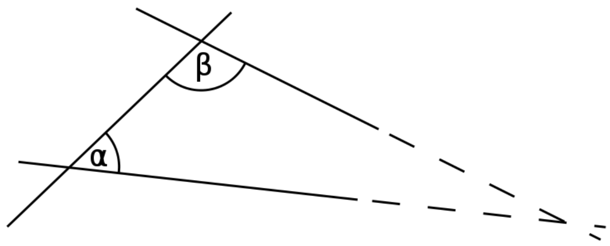
- All triangles have angle sum less than 180°.
- There is an upper limit to the area of a triangle.
- There are no similar triangles: if two triangles are the same shape, they are also the same size.
- There exist some triangles such that you cannot draw a circle passing through all three of their corners.

All these violate common sense, but no contradiction was in sight.

An Ancient Greek Conundrum

The issue is ancient, hinging on Euclid's *Parallel Postulate*:

If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.



(Image: Dickdock, Wikimedia Commons. Statement of postulate: Wikipedia. CC BY-SA 3.0. https://en.wikipedia.org/wiki/Parallel_postulate and https://en.wikipedia.org/wiki/File:Parallel_postulate_en.svg)

Playfair's axiom is a more intuitive equivalent: given a line and a point not on that line, there exists exactly *one* line through that point parallel to the original line.

Many geometers considered this too complicated to be a postulate: they thought it should be provable from Euclid's other axioms. Those were things that no one could see going wrong – like "A straight line can be extended indefinitely", "A circle can be drawn with any centre and radius", and "All right angles are equal".

But it turns out that *the existence of a Saccheri quadrilateral with summit right angles is equivalent to the truth of Euclid's parallel postulate*. You can't have one without the other. No wonder he couldn't prove it!

As another illustration, Euclid defines parallel lines as *lines that do not intersect*. But that doesn't forbid the possibility of *asymptotic* curves.

Think of the hyperbola $y = \frac{1}{x}$ and the axes – they never meet, but they get closer and closer. Surely these shouldn't count as parallel?

So Posidonius in the first century BC proposed to define parallel lines as *lines that stay the same distance from each other*. But without the Parallel Postulate you cannot prove that lines can do that!

Thus many attempts to prove the Parallel Postulate superfluous foundered upon the rocks over the ages: their authors at some point mentally substituted a natural-seeming thing that really assumed the Parallel Postulate all along.

But as the 19th century dawned, others had a brilliant thought: *what if the Parallel Postulate could not be proved?* Could one demonstrate a geometry in which it was *false*?

A Hungarian Breakthrough

				
Gauss	Bolyai	Lobachevsky	Beltrami	Poincaré

(Modern 2012 portrait of Bolyai by Ferenc Márkos, CC BY-SA 3.0. https://en.wikipedia.org/wiki/File:Bolyai_J%C3%A1nos_(M%C3%A1rkos_Ferenc_festm%C3%A9nye).jpg)

The sphere actualises the hypothesis of the obtuse angle. On it lines are the “shortest paths” – great circles. If we start at the equator, go a quarter-way round it to the east, go up to the North Pole, and then back down to where we started, we have traced out a gigantic triangle with three 90-degree angles.

So the angles of a triangle add up to *more* than 180°, those of a quadrilateral to *more* than 360°, and the summit angles of a Saccheri quadrilateral are obtuse.

Except: Euclid states that a line can be extended indefinitely in either direction. This is false on the sphere, where they curve back in on themselves. One can show that this must be true of any geometry where the hypothesis of the obtuse angle holds.

If we accept all but the Parallel Postulate of Euclid, we have *absolute* geometry. Saccheri was basically working here, so he could refute the obtuse angle.

But absolute geometry *cannot* decide between the right angle and the acute angle, though it can show that at most one of those hypotheses can hold. Either all triangles have an angle sum of 180°, or none of them do!

The breakthrough came from a Hungarian geometer. “Out of nothing I have created a strange new universe”, wrote János Bolyai to his father Farkas in 1823, enthusiastically describing his work on parallels. His father had dissuaded him from working on the problem three years earlier, but his son’s success changed his mind, and he published the work in 1832.

The great German mathematician Gauss, while praising this work, wrote to the younger Bolyai that he had had the ideas over thirty years earlier (but never published them)!

And later Bolyai learnt that Nikolai Ivanovich Lobachevsky had likewise produced a similar piece of work in Kazan, Russia, in 1829.

To honour these pioneers, the land of the acute angle is still called Lobachevskian geometry in Russia; in Hungary it is called Bolyai-Lobachevskian geometry.

An illustration of the Bolyai-Lobachevsky universe would serve as its final indisputable proof of legitimacy.

Unfortunately, there is nothing like the sphere: we cannot naturally embed hyperbolic geometry in Euclidean 3-space. Our "pocket hyperbolic universe" must somehow change the rules for distance and straightness.

The most famous such model was considered by the Italian mathematician Eugenio Beltrami. It is called the *Poincaré disc model*, because the French mathematician Henri Poincaré considered it fourteen years later, and more people paid attention to him than Beltrami.

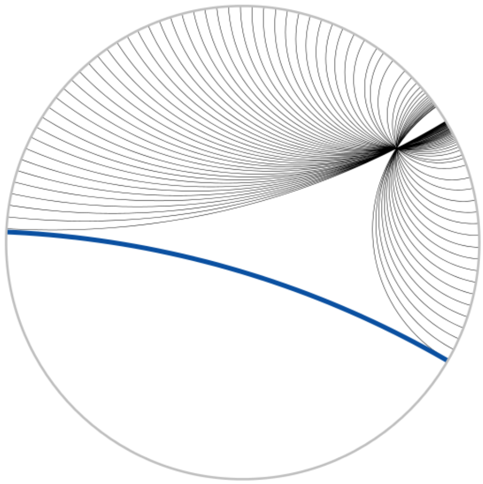
This "pocket universe" consists of just the points inside a circle of radius one.

Straight lines are line segments or circular arcs that intersect the boundary circle at right angles.

Distances become larger than they look as one approaches the boundary. If one started at the centre, one's first stride could perhaps take one halfway to the boundary. But the next stride would look a lot smaller. And so the next one, and the one after that; no matter how many strides one took, the boundary would remain forever out of reach.

Poincaré asks us to imagine a temperature gradient, with the hottest points in the centre and approaching absolute zero at the edge. A body's temperature at radius r is proportional to $1 - r^2$. This naturally rationalises these definitions, as objects expand when they heat up.

In this toy model the Parallel Postulate fails: given a line and a point not on that line, we can draw *as many lines as we please through that new point, all parallel to the original line!*



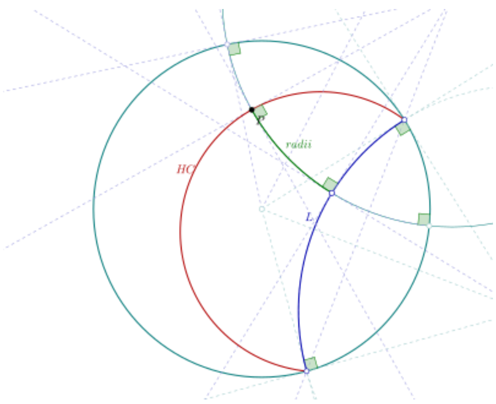
This technique of creating a “toy model” has been very fruitful, and not just in geometry. It entered set theory as Paul Cohen’s method of forcing. Here the galvanising problem was the existence of a set with more elements than the natural numbers, but fewer elements than the reals (Cantor’s *continuum hypothesis*).

Starting from a universe in which there aren’t many reals, one can “add some more” – like how we built a hyperbolic “toy universe” within our Euclidean one by changing the rules.

In the English language, the lands of the obtuse and acute angles are today called elliptic and hyperbolic geometry respectively. These names were given by the German geometer Felix Klein, by an analogy with conics.

Ellipses close in on themselves, as do lines in elliptic space. Hyperbolae go to infinity in both directions and have “two points at infinity”, as do lines in hyperbolic space. By this logic, Euclidean geometry is “parabolic”, though nobody says that.

In hyperbolic geometry, straight lines cannot be equidistant. The curve made by points equidistant from a given line *is not straight*. It is a *hypercycle* – in some sense, a circle with “greater than infinite” radius.



As a circle attains infinite radius, it becomes a *horocycle,* tangent to the boundary. Expand it further still, and we get a hypercycle, and eventually a straight line. In Euclidean geometry a circle already becomes a line at “infinite radius”; in spherical geometry, there are only circles!

Hyperbolic straight lines can intersect; be *limit parallel* or asymptotic, “meeting at the boundary”; or be *ultraparallel* and never even intersect at the boundary, having always a minimal degree of separation.

In spherical geometry, all lines intersect; in Euclidean geometry, they can either intersect or be parallel. In *projective geometry* – roughly, the geometry of perspective – one can rationalise the latter as “intersecting at infinity”.

Many such “analogies” link the three geometries!

Your Family Tree and other Biological Applications

Does hyperbolic geometry have uses beyond questioning our assumptions?

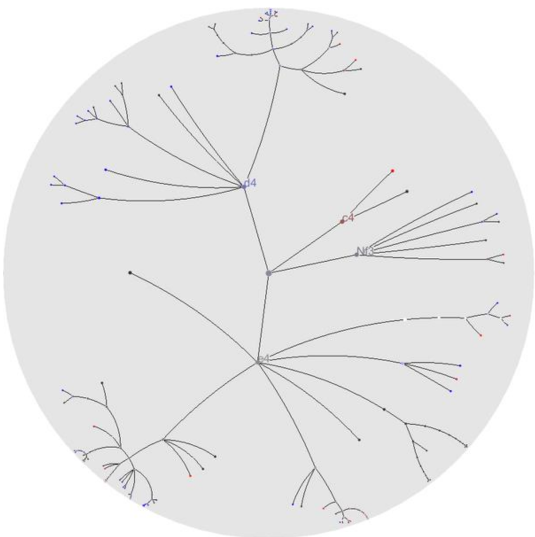
Its exponentially growing nature provides many!

In Euclidean space, the area of a circle of radius r is πr^2. In spherical geometry, it goes like a trigonometric function: $4\pi \sin^2 \frac{r}{2}$.

Naturally, in hyperbolic geometry it goes like a hyperbolic function: $4\pi \sinh^2 \frac{r}{2}$. That is an exponential!

Many natural processes exhibit exponential growth. Imagine drawing your family tree on paper. We have two parents, and each of them have two parents, and so did each of them – and this would spiral off the page within a few generations (and we didn’t even consider siblings, cousins, uncles, and aunts).

But on a hyperbolic sheet of paper, there’d be no problem!



What would a surface spiralling out with so much area look like? Very crinkly, like a fractal!

Whenever Nature wishes to maximise surface area – leaves of a tree trying to find light, or lungs trying to get oxygen – we find surfaces naturally admitting hyperbolic geometry! Fractal phenomena in Euclidean geometry are like "echoes" of hyperbolic geometry!

There is even a precise sense in which *most* abstract surfaces in any number of dimensions naturally admit hyperbolic geometry!

The Geometry of the Universe

Is our universe hyperbolic? Current measurements are not precise enough to distinguish between the three geometries.

But it has *local* regions of curvature. Think of a piece of paper – it is flat, with zero curvature. Push its ends together, keeping them flat on a table. The top now looks like a dome – positive curvature (like spherical geometry) has accumulated there. Curvature is conserved, so this is cancelled by regions of negative curvature (like hyperbolic geometry), where the paper leaves the table.

Einstein's general relativity tells us that *concentrations of matter curve space*. Think of the depression that appears on a trampoline when a bowling ball is put on it. Therefore the study of geometries that are not Euclid's plays a significant role in helping us understand our own universe at the largest scales.

The geometry close to the Sun, clearly a large concentration of matter, is not Euclidean. The first sign of this came when French astronomer Urbain Le Verrier (discoverer of Neptune) noticed that Mercury was not quite orbiting as it should be.

Here we reach *Riemannian* geometry, which generalises these first non-Euclidean geometries and allows for non-constant curvature.

So not only did hyperbolic geometry help free us from misconceptions, but it also taught us a lot about the natural world!