The Secret of Manholes

Hyunseo Chang

INTRODUCTION

Why are manhole covers round? What happens if we use squares or triangles instead? The key is in the constant width. one of the properties of a circle. A circle is a two-dimensional curved shape, consisting of all points that are equidistant from a given point, the center so, by definition, a circle has a constant width. This means a circle always has the same width regardless of the part being measured which prevents circle manhole lids from passing through the circle manhole and falling in any direction or position. However, shapes that do not have constant width work differently. For example, squares have different widths depending on which way the width is being measured. As shown in Fig. 1, for a square with side x, the distance between opposite sides is x while the distance between opposite vertices, i.e., the length of the diagonal, is $x\sqrt{2}$ (based on the Pythagorean theorem) which is longer than x. In fact, more generally, all polygons do not have constant width so when a polygon manhole lid gets positioned in the wrong direction, because of the width difference by parts, there could be an occasion where the manhole lid has a smaller width than the width of the hole. Therefore, there is a chance of the lid falling through the polygon hole, which is why, only circles or curves with constant width, are used to make manholes.

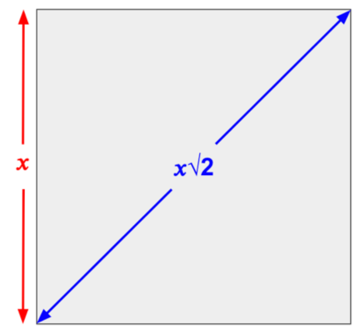
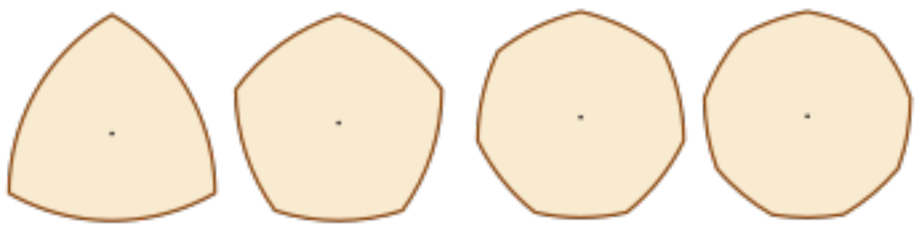


Fig. 1

In this essay, another special type of curve of constant width and the theorems associated to these constant width curves.

REULEAUX POLYGONS

Have you ever heard of the word 'Reuleaux'? As its special name, reuleaux polygon has a unique shape that you may never have seen before:



They look like circles, but at the same time look like polygons. What are these geometric shapes and how are they produced? Reuleaux polygon is a "plane convex figure of constant width w whose boundary consists of a finite (necessarily odd) number of circular arcs of radius w ". In other words, it is a curve of constant width that is made up of circular arcs of constant radius equal to the width of the curve. Reuleaux polygons can be constructed from regular polygons with an odd number of sides or from certain irregular polygons by connecting two adjacent vertices by a circular arc centered on the third vertex. This third vertex is on the opposite vertex from the two adjacent vertices. The construction method explains why reuleaux polygons have a shape of a mixture of a circle and a polygon.

BARBIER'S THEOREM

Barbier's theorem states that every curve of constant width, regardless of shape, has a perimeter equal to its width multiplied by π. This means that different types of curves of constant width with the same width has the same perimeter, which is $\pi * width$. This theorem is closely related to circles and reuleaux polygons which have constant widths and can be a powerful tool that shows different shapes with the same perimeters. There are various methods to prove this theorem, such as using the Minkowski sum or Crofton formula, but for a better understanding, I made a different approach and established a more comprehensive proof focusing on the properties of constant width curves.

PROOF

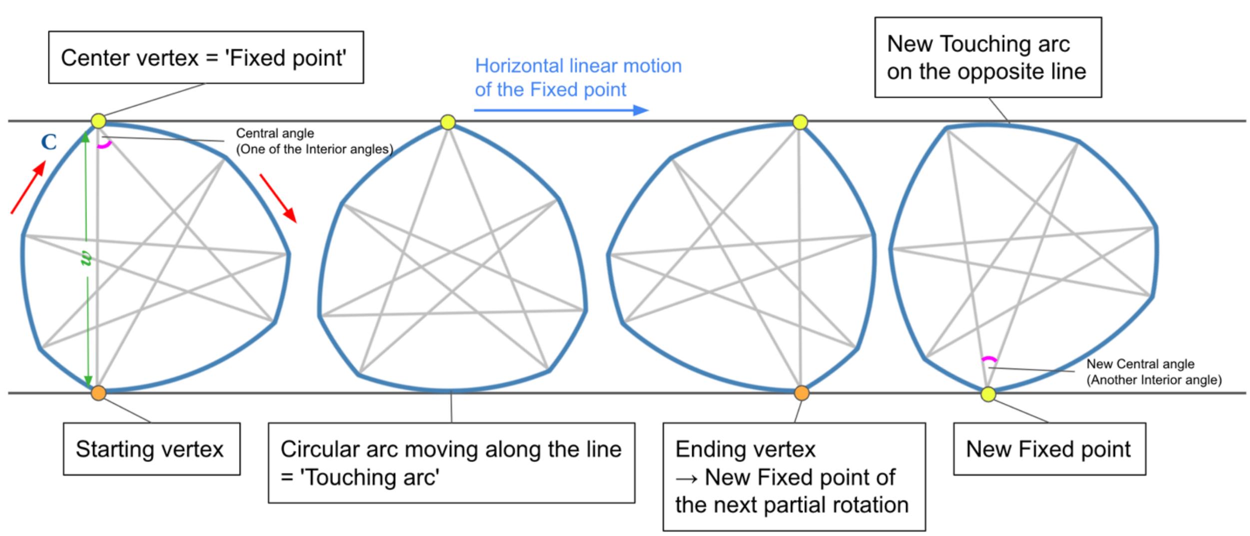
Let there is a curve of constant width w placed between two parallel lines which are apart from each other by distance w, the width of the curve. To calculate the perimeter, in other words, the sum of the length of all circular arcs, the shapes will be rolled along between the parallel lines, and the trace of the arcs will be evaluated.
When calculating the total distance traveled (or the trace) in one complete rotation of 2π, the method is different for reuleaux polygons and circles.

1. Circle
For a circle, all parts move together with the curve tangent to the parallel lines, so the total distance traveled in one rotation is equal to the circumference.

$$r = \frac{w}{2}$$

$$Perimeter = Circumference = 2\pi r = 2\pi * \frac{w}{2} = \pi w$$

2. Reuleaux polygons
However, reuleaux polygons travel in a slightly different way.



Let C be a reuleaux polygon that has a constant width w and is placed between the same two parallel lines. The circular arcs made up of C are constructed with the radius w about the center at each vertex. The rotation can be broken down into multiple partial rotations focusing on each movement made by each circular arc. During a partial rotation, the circular arc which is one side of C will move along one line and the corresponding center vertex will be fixed on the opposite line and travel in a horizontal linear motion. Let's call the circular arc that C is moving along the 'touching arc' and the corresponding center vertex the 'fixed point'.

The angle of each partial rotation is equal to the central angle of the touching arc, which is one of the interior angles. Each partial rotation will keep going from the starting vertex until the ending vertex of that touching arc, and a new partial rotation will start using the ending vertex as the new fixed point. Then the opposite circular arc becomes the new touching arc but now traveling along the other line compared to the previous partial rotation. This means C travels along the upper and lower lines alternatively.

During one complete rotation of C, each circular arc side meets the lower line once and the upper line once, therefore twice in total. This means each interior angle is counted twice in a 2π rotational movement, so the total sum of interior angles will equal π.

$$2 * (Total\ sum\ of\ interior\ angles) = 2\pi$$

$$Total\ sum\ of\ interior\ angles = \pi$$

The perimeter of C is equal to the total sum of its circular arcs which has the same radius w, and if the sum of circular arcs was drawn into a single circular arc, based on the sum of interior angles, it can be deduced that the central angle of that single circular arc will be π. Therefore, the perimeter of C will be equal to $\pi * w$:

$$Perimeter = \pi * w$$

ISOPERIMETRIC INEQUALITY

Isoperimetric itself means having the same perimeter, which can be considered an extension of Barbier's theorem. Based on Barbier's theorem, it has been shown that all constant width curves, including circles and reuleaux polygons with the same width, have the same perimeter, i.e., are isoperimetric.

The isoperimetric inequality of a plane, which is in two-dimensional space, states, for the length L of a closed curve and the area A of the planar region enclosed by the closed curve:

$$L^2 \geq 4\pi A$$

Although isoperimetric inequality can be proved in various ways, in this paper, we will use the Proof by E. Schmidt (1939).

PREPARATION:

We will parametrize the curves by arc length which means the particle moves along the curve at a constant rate of one unit per second. x and y will be expressed in terms of parameter t. Integration of x in respect of y to find the area below the curve:

$$\begin{aligned}\int x\,dy &= \int x(t)\frac{dt}{dt}\,dy \\ &= \int x(t)\frac{dy}{dt}\,dt \\ &= \int x(t)y'(t)\,dt\end{aligned}$$

Integration by parts is used to find an alternative expression of the integration. Since $c(t)$ is a closed curve, the start point and endpoint i.e. the lower and upper boundaries of integration are the same. Parameterized by arc-length with $t \in [a, b]$, we have that $c(a) = c(b)$:

$$\begin{aligned}\int_a^b xy'\,dt &= \int_a^b (xy)'\,dt - \int_a^b x'y\,dt \\ &= [x(b)y(b) - x(a)y(a)] - \int_a^b x'y\,dt \\ &= -\int_a^b x'y\,dt\end{aligned}$$

Cauchy-Schwarz Inequality:

$$(ab - cd)^2 \le (a^2 + c^2) * (b^2 + d^2)$$

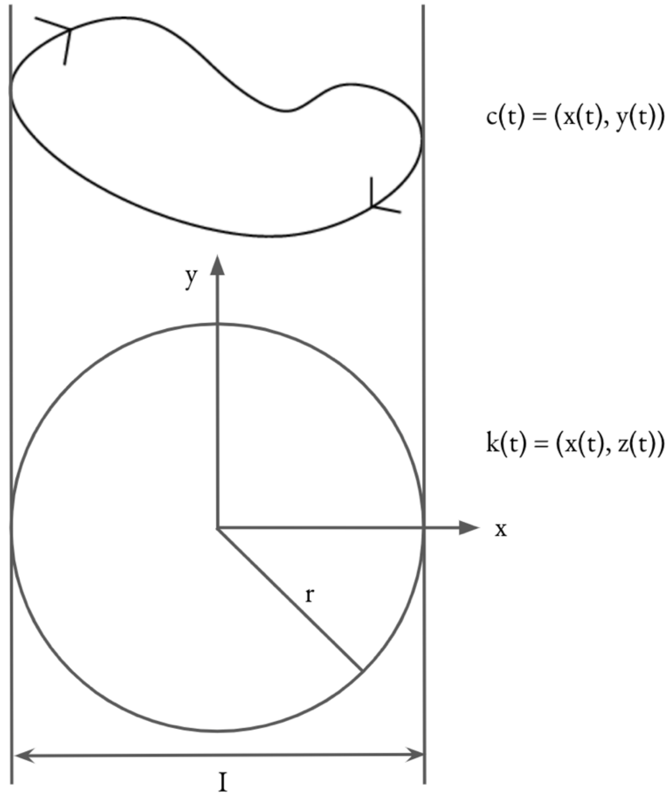
Proof.

$$\begin{aligned}&(a^2 + c^2) * (b^2 + d^2) - (ab - cd)^2 \\ &= a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 - a^2b^2 - (-2abcd) - c^2d^2 \\ &= a^2d^2 + c^2b^2 + 2abcd \\ &= (ad + cb)^2 \ge 0\end{aligned}$$

Therefore,

$$(ab - cd)^2 \le (a^2 + c^2) * (b^2 + d^2)$$

PROOF:



$$I = [-r, +r]$$

$$x(t) \in I$$

$x(t)$ in $k(t) = (x(t),\ y(t))$ is same to the $x(t)$ in $c(t)$, meaning $x(t)$ of $k(t)$ has the same parameterization as in $c(t)$.

Let's find the areas of the closed curve, $c(t)$, and circle, $k(t)$, to prove the isoperimetric inequality. The lower and upper boundaries of the integral are 0 and L, the perimeter of $k(t)$ because $k(t)$ is parameterized by the arc length which means the particle moves one unit per second, so the time taken for the particle to move around the circle is

$$t = \frac{s}{v} = \frac{L}{1} = L$$

Therefore, the particle for integration should move from $t = 0$ to $t = L$:

$$A = Area\ of\ c(t)$$
$$B = Area\ of\ k(t) = \pi r^2$$

$$A = \int_0^L x(t)y'(t)\,dt$$

$$B = \int_0^L x(t)z'(t)\,dt = -\int_0^L z(t)x'(t)\,dt = \pi r^2$$

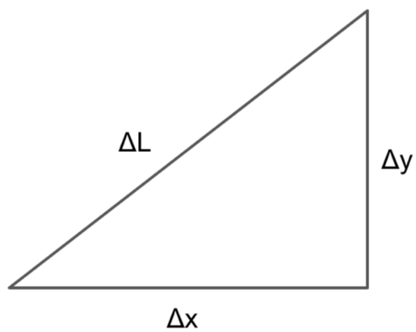
$$A + B = A + \pi r^2 = \int_0^L xy'\,dt - \int_0^L zx'\,dt = \int_0^L (xy' - zx')\,dt$$

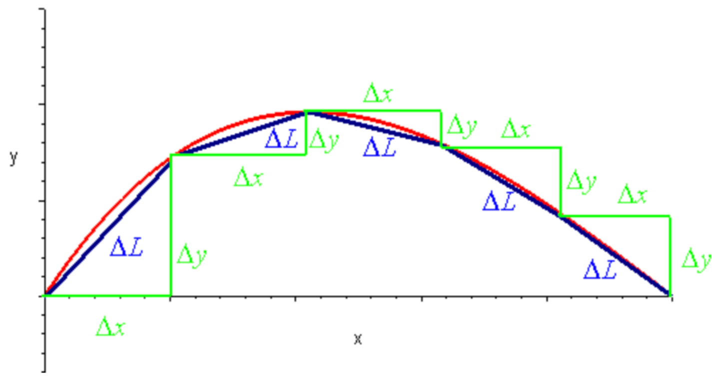
$$(xy' - zx')^2 \leq (x^2 + z^2) * ((y')^2 + (x')^2)$$

Let's use the Cartesian equation of circle $k(t)$: $x^2 + z^2 = r^2$ (1)

As curve $c(t)$ is parameterized by arc length, which means the particle travels by one unit per second along the curve. This means:

$$\frac{dL}{dt} = 1$$





$$(\Delta L)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\left(\frac{\Delta L}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2$$

$$\left(\frac{dL}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\begin{aligned}(y')^2 + (x')^2 &= (x')^2 + (y')^2 \\ &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\ &= \left(\frac{dL}{dt}\right)^2 \\ &= 1^2 \\ &= 1\end{aligned}$$

(2)

By (1) and (2),

$$(xy' - zx')^2 \leq r^2 * 1$$
$$(xy' - zx')^2 \leq r^2$$

If we substitute this back to the equation of $A + B$:

$$A + \pi r^2 = \int_0^L (xy' - zx')dt$$
$$= \int_0^L \sqrt{(xy' - zx')^2}\, dt$$
$$\leq \int_0^L \sqrt{r^2}\, dt$$
$$= Lr$$

(3)

Let's use the Inequality of Geometric and Arithmetic mean:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

(Where $a, b \in \mathbb{R}^+$, with equality when $a = b$)

Use $a = A$ and $b = \pi r^2$ for the Inequality of Geometric and Arithmetic mean:

$$\frac{A + \pi r^2}{2} \geq \sqrt{A * \pi r^2}$$

Substitute Lr from (3):

$$Lr \geq 2\sqrt{\pi A r^2}$$

Square both sides:

$$L^2 r^2 \geq 4\pi A r^2$$
$$L^2 \geq 4\pi A$$

Thus, the Isoperimetric Inequality is proved.
Based on the Inequality of Geometric and Arithmetic mean, the equality is satisfied when

$$A = \pi r^2$$

which is when the closed curve is a circle.

From the Isoperimetric inequality, it can be shown that with a given perimeter, the maximum area can be obtained by a circle.

CONCLUSION:

Let's combine the concept of Isoperimetric inequality with Barbier's theorem. Curves of constant width that have the same width, w have equal perimeters of πw regardless of their shapes. If there are figures with equal perimeters, by the isoperimetric inequality, it is shown that the circle has the largest area.

Therefore, reuleaux polygons and circles of the same width have the same perimeter but different areas. Circles have the largest area and reuleaux polygons have smaller areas. The area of the reuleaux polygon decreases as the number of vertices decreases. This means if there is a reuleaux triangle, pentagon, and heptagon, then the reuleaux triangle has the smallest area, the reuleaux pentagon has the second largest area, and the reuleaux heptagon has the largest area. In conclusion, the closer the geometrical shape is to a circle, the larger the area it will have.

Reuleaux polygons have very interesting properties and are used in various areas. For example, some manholes are made in the shape of reuleaux triangles because they can perform the same as circle manholes can. As they are constant width curves like circles, the manhole lid cannot fall through the hole. Also, shown by Barbier's theorem and the Isoperimetric inequality, reuleaux polygons have smaller areas than circles with the same width so it can reduce the cost and resources being used in manholes.

In addition, Reuleaux polygons can add aesthetics to designs as they have a unique shape that gives the feeling of straight lines and curves at the same time. The image on the left is a reuleaux triangle-shaped window in the Church of Our Lady, Bruges. The shape gives a sense of stability and softness together that no other shape can do.

Finally, it could be an alternative geometry used for wheels. Reuleaux polygons have a constant width, so their rotation is similar to how a circle rotates. When it is placed between two parallel lines separated by its constant width, while rotating, the vertices do not exceed the lines and keep touching both lines. This allows its application to wheels or tires, and its properties that are similar to polygons can be helpful when going up the stairs, adding mobility to people with disabilities or walking difficulties.