Buffon's Needle: An Expedition into Geometric Probability

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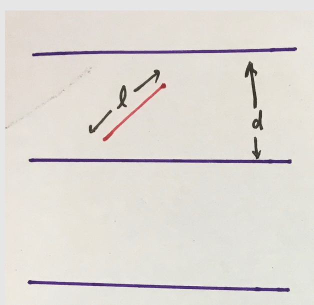
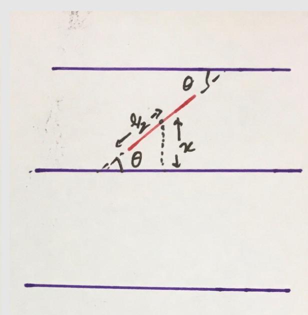
Probability Density Functions

#1 What is the probability that a needle randomly dropped on a surface etched with equally spaced parallel lines will intersect one of the lines?

Titled "Buffon's Needle," this is one of the earliest problems posed in the field of geometric probability. Comparative measurements, instead of counting discrete, equally probable events, form the foundation of problems in geometric probability. Problems of this kind also birthed integral geometry, an incredibly useful tool in geometric probability theory. Before proceeding, I strongly advise you to grab a pencil and paper.

Buffon's Needle may seem dauntingly abstract. However, readers can make some simple observations. Visualize yourself throwing a needle on a tiled floor, or you can even try the experiment yourself! Does your needle intersect any lines? If so, does it intersect two lines or one? This simple visualization brings us to our first observation: The length l of the needle can be either shorter than, equal to or longer than the distance d between two consecutive parallel lines. A needle *must* be longer than d for it to be able to intersect two lines. We shall discuss the former scenario, informally referred to as the "short needle" problem, where $l < d$. This needle either intersects one line or none. *It cannot intersect two lines.*





What variables can we define? Consider the midpoint of the needle at $l/2$. Drop a perpendicular from this point to the parallel line closest to it. Let the length of this perpendicular be x. Since this perpendicular is drawn to the line closest to the needle, x cannot exceed half of d. Notice how the needle makes two angles with the parallel lines:one acute and one obtuse. We define θ to be the *acute angle* the needle makes with the parallel lines. These two conditions aptly construe a randomly thrown needle.

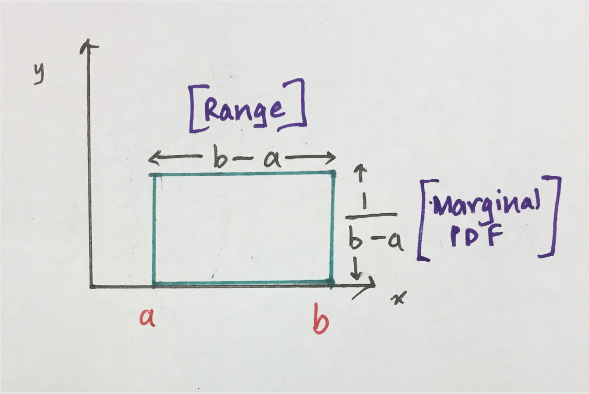
$$0 \leq x \leq \frac{d}{2}$$
$$0 \leq \theta \leq \frac{\pi}{2}$$

How does one *define* sample spaces for such "abstract" problems? Think about it. Our conditions on x and θ are quite straightforward. But what if we were dealing with points on the complex plane? Or on uneven solids? Such problems beget the use of complex integral geometry but the core idea remains the same : define variables and assign conditions to them. In this example, the sample space is simply the set of all ordered pairs(x, θ) that satisfy the two inequalities above.

On to constructing a probabilistic model. For this we utilize *probability density functions*. To illustrate, consider a football match. The typical football match is 90 minutes long. What is the probability that a player will score exactly on the 37^{th} minute? Many players score approximately 37 minutes into a match, but the chance that a player scores at exactly t=37.0000.... minutes is 1/infinity, or zero. However, we *can* quantify the probability she scores *between* 37 and 37.5 minutes. Assume this probability is 0.05. Then the chance of scoring between 37 and 37.005 minutes is 0.0005 since this interval is one-hundredth the original. Between 37 and 37.00005 minutes it is 0.000005. And so on. Notice how the ratio $\frac{probability of scoring in an interval}{duration of the interval}$is constantly about 5 per minute. This quantity, $5min^{-1}$, is the probability density of scoring at around 37 minutes. It follows that the probability of scoring within an infinitesimal time period of duration dt around 37 minutes, is $(5min)^{-1}\,dt$. Therefore, there exists a probability density function f with $f(37minutes) = 5min^{-1}$. Integrating this function over any window of time yields the probability that the player scores in that period. Naturally, the integral over the entire duration of the match is then 1.

Formally,

$$P\left(a \leq k \leq b\right) = \int_a^b f(k)dk$$



x and θ have predefined ranges with uniformly distributed potential values. They are examples of independent *discrete random variables.* Let us assume that k is a discrete random variable. with a predefined range [a, b]. k can geometrically be represented as a rectangle of length *b-a*, which represents its range. The height of the rectangle is $\frac{1}{b-a}$,, the *marginal*, or individual probability density function. The area of the rectangle, which represents the total probability of k falling in range $[a, b]$ is then 1 square unit, representing 100 percent, thus confirming our notion. Therefore the individual probability density function of k, termed its *marginal probability density function*, is:

$$f(k) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq k \leq b \\ 0 & \text{if } k \leq a \text{ or } k \geq b \end{cases}$$

Computing marginal probability density functions for x and θ over their predefined ranges:

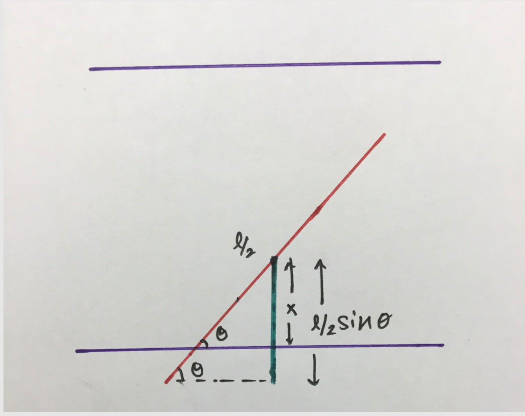
$$f_{\mathrm{x}}(x) = \frac{2}{d}$$

$$f_{\theta}(\theta) = \frac{2}{\pi}$$

We may now construct a joint probability density function to define sample space probability. Without delving too deep into probabalistic theory:Since x and θ are independent variables, their joint probability density function is simply the product of their marginal probability density functions. This makes intuitive sense.

Therefore,

$$f(x) = \frac{4}{\pi d}$$



When does the needle intersect a line? We observe that the green line, the "vertical extent" of the triangle below must be longer than distance x. This vertical extent is clearly $\frac{l}{2} sin\theta$. Therefore, for a needle to intersect a line: $x < \frac{l}{2} sin\theta$.

Integrating x to $\frac{l}{2} sin\theta$ and θ to $\frac{\pi}{2}$to find joint probability,

$$\text{P(a needle crosses a line)} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l sin\theta}{2}} f(x) dx d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{l sin\theta}{2}} \frac{4}{\pi d} dx.d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{4}{\pi d} \cdot \frac{l}{2} sin\theta . d\theta$$

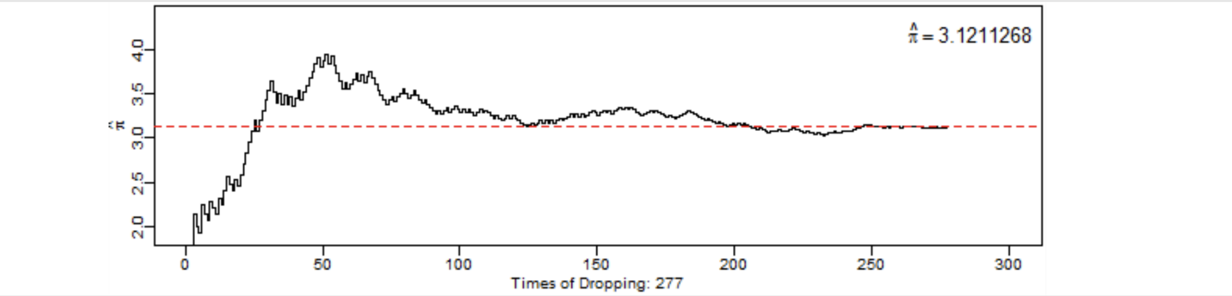
$$\left[\frac{2l}{\pi d}(-cos\theta)\right]_0^{\frac{\pi}{2}}$$

$$\text{P(a needle crosses a line)} = \frac{2l}{\pi d}$$

And voila, we have computed the "short" needle's likelihood of intersecting a line. Notice how in doing so we have also procured a fascinating method to compute π!

$$\pi = \frac{2l}{\text{P}(aneedlecrossesaline).d}$$

Here is a graph of experimental values of π plotted against its actual value after dropping the needle 277 times:



Larger sample sizes, accompanied by intensive random sampling, termed *Monte Carlo simulation*, will yield approximations closer to π Although developing a formula for π was not an intended consequence of this problem, scientists and engineers do frequently use Monte Carlo methods to approximate important scientific constants. Mathematical labour rarely goes to waste.

Now that we have a general outline to solve problems in geometric probability, here is a slightly more complex problem:

#2 Solve Buffon's Needle for the "long needle" scenario where $l > d$. Try to construct suitable probability density functions!

Total Expectation Theorem

Consider a log of length l. Suppose a lumberjack chops this log at X. The left side of the broken log is then chopped at Y. Again the left piece is kept. What is the *expected* length of this piece of the log?

In probability theory expected value refers to the average value of potential outcomes.

Length of the log remaining after the first chop, x, is uniformly distributed between $[0, l]$. Length of the log remaining after the second chop, y, now uniformly varies between $[0, x]$.

$$0 \leq \ y \leq x \leq l.$$

Unlike the previous example x and y are not independent variables. The range of potential values of y is *conditional* on the value of x. We can now define a marginal probability distribution function for x and a *conditional probability distribution function* for y:

$$f_x(x) = \frac{1}{l}$$

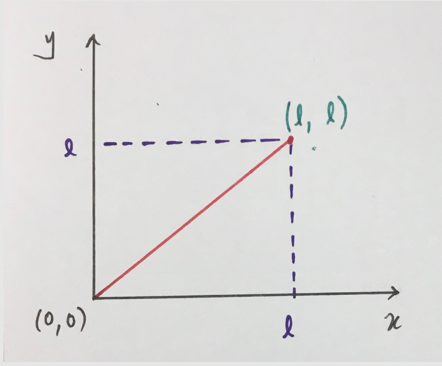
$$f_{y|x}(y|x) = \frac{1}{x}$$

Utilizing the multiplication rule to compute a joint probability density function:

$$f_{xy}\,(x, y) = f_x(x).f_{y|x}\,(y|x) = \frac{1}{l}.\frac{1}{x}$$

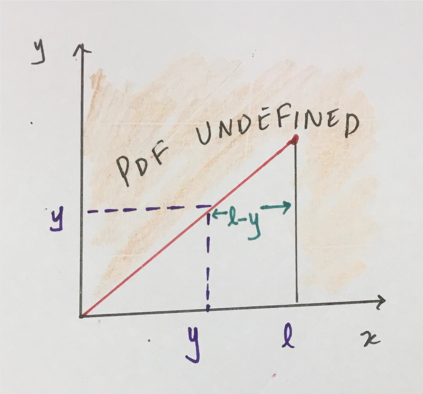
$$f_{xy}\,(x, y) = \frac{1}{lx}$$

We can visualize the function as follows:



Notice how the slope of this line is 1. From $0 \leq \ y \leq x \leq l$ the maximum value of both x and y is l. The topmost vertex corresponds to (l, l).

We can now find the marginal probability density function for y by integrating the joint probability density function over the range of values of x. What is this range? For a given value of y the joint probability density function is defined only for values of x between x=l and x=y.



$$f_y(y) = \int_y^l f_{xy}\left(x, y\right) dx$$

$$\int_y^l \frac{1}{lx} dx$$

$$f_y(y) = \frac{1}{l} log\left(\frac{l}{y}\right)$$

The expected value of Y can by found by integrating y times its probability density. The range of integration is all possible values of y, $[0, l]$.

$$E\left[y\right] = \int_0^l y.\frac{1}{l} log\left(\frac{l}{y}\right) dy$$

A complex integral indeed! Should we take the algorithmic route and proceed to integrate by parts? Or maybe there is a simpler alternative? Mathematics begets elegance... Of course we should search for a simpler approach!

The *Total Expectation Theorem* comes to our rescue. The expected value of an experiment can be calculated by a weighted method: by adding the products of the probabilities of events and *their* expected values.

$$E\left(A\right) = P\left(K_1\right) E\left(A|K_1\right) + P\left(K_2\right) E\left(A|K_2\right) + ... + P\left(K_n\right) E\left(A|K_n\right)$$

x has infinite potential lengths between 0 and l. Instead of summing discrete weights we must integrate the product of probability density of x ($\frac{1}{l}$) and the expected value of y given x over this range ($E\left(y|x\right)$).

$$E\left(y\right) = \int_0^l \frac{1}{l} E\left(y|x\right) dx$$

where

$$E\left(y|x\right) = \frac{x}{2}$$

$$E\left(y\right) = \int_0^l \frac{1}{l}.\frac{x}{2} dx$$

$$E\left(y\right) = \frac{1}{2} E\left(x\right)$$

$$E\left(x\right) = \frac{l}{2}$$

$$E\left(y\right) = \frac{1}{2}.\frac{l}{2} = \frac{l}{4}$$

This is an intuitive answer. After the first chop we can expect the mean value of the log length to be $\frac{l}{2}$. Similarly for the second chop we obtain an expected value of $\frac{l}{4}$. *"I could have guessed that!"* Now you have the probabalistic logic to prove it! This is not to say that computational thinking in any way subdues intuition. Intuitive thinking is an immensely powerful tool in a mathematical environment. Many complex problems can be greatly simplified via intuition. After all, algorithms can be encoded- cognition sets us apart. A good example of a problem requiring probabilistic intuition, especially because its mathematical proof is rigorous, is problem A6 from the 1992 Putnam Competition:

#3 Four points are chosen uniformly at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points?

Problems

Unchallenged? Here are some practice problems! For an even greater challenge solve using probability density functions.

#4 Three points are randomly chosen on a circle. What is the probability that the triangle with the vertices at the three points has the center of the circle in its interior?

#5 Assume a stick is broken at random into three pieces. What is the probability that the pieces can form a triangle?

Bibliography

All related pages from (https://www.cut-the-knot.org) The image of the Buffon's Noodle simulation is from (http://www.di.fc.ul.pt/ jpn/r/animation/buffon.needle.html). All other images are my own.