Quaternions: How 4-dimensional complex numbers are used for computer graphics

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Gimbal lock - if you play a lot of video games, chances are, you've probably experienced it before. It happens when there is a loss of one degree of freedom in a three-dimensional mechanism which locks the system into rotation in a two-dimensional space. This can cause some unusual looking animations and rotations in computer graphics.

This issue affects the real world too. If you have a camera which can physically rotate in 3 dimensions on a gimbal, you may experience gimbal lock if 2 of the axes rotate too close to each other. An incident of gimbal lock occurred on the Apollo 11 mission, where a set of 3 gimbals were used on an inertial measurement system, instead of 4.

In computer graphics, the solution to this problem is to use quaternions to calculate 3D rotations. Quaternions are 4-dimensional complex numbers which were invented by William Hamilton in 1843. They provide a convenient way of describing the orientation of an object in a three dimensional space, which is especially useful for computing graphics for video games.

1. History of Quaternions

1.1. Wessel and Argand - Invention of the complex plane

The complex plane is a 2 dimensional plane which enables us to visualise complex numbers in a Cartesian coordinate system by using the x-axis to represent the real part and the y-axis to represent the imaginary part. The complex plane is also commonly known as the Argand diagram.

In 1806, the amateur Swiss mathematician Jean-Robert Argand wrote a paper on his idea of a geometric representation of complex numbers [6]: "*Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*". However, this paper was published privately, without his name and was not distributed widely. The mathematician Legendre received a copy of the work from Argand and sent it to François Français in 1806. After François Français's death in 1806, the paper was rediscovered by his brother, Jacques Français, who republished Argand's ideas in the journal "*Annales de mathématiques*". Argand came forward as the original author and submitted a slightly modified version of his original work to the journal.

However, Argand was not the first to discover the complex plane. A Norwegian surveyor named Caspar Wessel was the first to publish this geometric interpretation in 1799, after submitting his paper to the Royal Danish Academy of Sciences. This paper remained hidden from the mathematical community, including Argand, for nearly a century and was only discovered in 1895. Although most mathematicians agree that Wessel was the first to discover the complex plane, it is still named after Argand to this day.

1.2. William Hamilton - Triples and Quaternions:

The existence of complex numbers presented a question for mathematicians of the 19th century. Could there be a generalisation of two-dimensional complex numbers to three dimensions?

Hamilton set out to solve this problem with the following 2 requirements [1]:
1) It is possible to multiply out complex numbers term by term
2) The length of the product of the vectors must be equal to the product of the lengths. This was also called the "law of the moduli" by Hamilton

After Hamilton, Hurwitz proved that these 2 requirements can only be fulfilled in 1, 2, 4 and 8 dimensions. So Hamilton's attempt in three dimensions would not work.

Initially, Hamilton supposed that as a 2D complex number is represented by $x + iy$, a 3D complex number could be represented by the triple $x + iy + jz$, where i and j are imaginary numbers and both square to -1.

However, a triple in this form fails to close when squared:

$$\begin{aligned} z &= x + iy + jz \\ z^2 &= (x + iy + jz)(x + iy + jz) \\ &= x^2 + ixy + jxz + ixy - y^2 + ijyz + jxz + jiyz - z^2 \\ &= x^2 - y^2 - z^2 + 2ixy + 2jxz + 2ijyz. \end{aligned}$$

There should be 3 dimensions to this number:
- A real part: $x^2 - y^2 - z^2$
- An imaginary part, i, with coefficient $2xy$
- An imaginary part, j, with coefficient $2xz$

The term $2ijyz$ seems to be excessive and so the triple does not close when squared. In his notebook, Hamilton remarked:

> *Its real part ought to be $x^2 - y^2 - z^2$ and its two imaginary parts ought to have for coefficients 2xy and 2xz; thus the term 2ijyz appeared de trop, and I was led to assume at first ij = 0. However I saw that this difficulty would be removed by supposing that ji = −ij.* [2]

Although Hamilton's triples refuse to close when squared, they can be easily added or subtracted. In a letter to his son Archibald, Hamilton wrote:

> *Your brother William Edwin and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."* [2]

Hamilton tried to solve this problem for over a decade. He introduced a third imaginary term *k* such that *k = ij.* On October 16, 1843, Hamilton was walking with his wife along the Royal Canal in Ireland when he came up with his solution, which he engraved into the stone of Broome Bridge [5]:

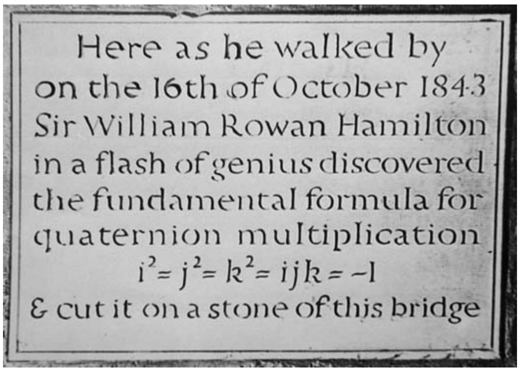

Figure 1 – A plaque commemorating Hamilton's discovery

If the conditions $i^2 = j^2 = k^2 = ijk = -1$ were postulated, a solution could be found. This would imply that $ij = -ji$, a non-commutative law of multiplication. This led Hamilton to define the solution as a quaternion, in the form $z = a + bi + cj + dk$. [3]

2. Quaternion Arithmetic

This section presents the rules of quaternion arithmetic with which calculations can be performed later on.

2.1. Defining a Quaternion

There are multiple ways to define a quaternion. Here are 3 3 commonly used definitions:

1) $q = s + xi + yj + zk$

   This is Hamilton's original definition of a quaternion with a real part and all 3 imaginary terms.

2) $q = s + \mathbf{v}$

   A quaternion can be also defined as the addition of a scalar part and a vector part, where the scalar part refers to the real part and the vector part refers to the imaginary part, $xi + yj + zk$

3) $q = [s, \mathbf{v}]$

   A quaternion can be also represented as an ordered pair comprising a scalar part and a vector part. This is known as the algebraic definition.

A quaternion must follow the following obligatory rules:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k, \quad jk = i, \quad ki = j$$
$$ji = -k, kj = -i, ik = -j.$$

2.2. Real and Pure Quaternions

A real quaternion is a quaternion with a zero vector part:

$$q = [\, s \,,\, 0 \,]$$

It behaves just like a real number as it is just a scalar. The product of two real quaternions is another real quaternion:

$$q_a = [\, s_a \,,\, 0 \,]$$
$$q_b = [\, s_b \,,\, 0 \,]$$
$$q_a q_b = [\, s_a s_b \,,\, 0 \,]$$

A pure quaternion is a quaternion with a zero scalar part, meaning that it is just a vector with imaginary parts:

$$q = [\, 0 \,,\, \mathbf{v} \,]$$

$$q = xi + yj + zk$$

The product of two pure quaternions is shown below:

$$q_a = [\, 0 \,,\, \mathbf{v}_a \,]$$
$$q_b = [\, 0 \,,\, \mathbf{v}_b \,]$$
$$q_a q_b = [\, 0 \,,\, \mathbf{v}_a \,] [\, 0 \,,\, \mathbf{v}_b \,]$$
$$q_a q_b = [\, -\mathbf{v}_a \,.\, \mathbf{v}_b \,,\, \mathbf{v}_a \times \mathbf{v}_b \,]$$

The product of two pure quaternions is no longer pure as the real part is the negative dot product.

2.3. Complex Conjugates of Quaternions

Every complex or hypercomplex number has a complex conjugate, which is the number with an equal real (scalar) part and an imaginary (vector) part equal in magnitude but opposite in sign.

The conjugate of a quaternion, q, is normally given as q^*:

$$q = [\, s \,,\, v \,]$$
$$q^* = [\, s \,,\, - v \,]$$

The product of a quaternion and its conjugate is a real number:

$$qq^* = [\, s \,,\, v \,]\,[\, s \,,\, - v \,]$$
$$qq^* = [\, s^2 + v^2 \,,\, 0 \,]$$

Example:

$$q = [1, - 2i + 3j - 4k]$$
$$q^* = [1, + 2i - 3j + 4k]$$

$$qq^* = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

The complex conjugate can be used to work out the inverse of a quaternion, as shown in Section 2.6.

2.4. Norms and Normalisation of Quaternions

The norm of a quaternion is the absolute length of the quaternion, also known as its magnitude. $|q|$ is used to denote the norm of a quaternion, q.

This can be calculated as follows:

$$q = s + xi + yj + zk$$
$$|q| = \sqrt{(s^2 + x^2 + y^2 + z^2)}$$

Alternatively, this can be represented using an ordered pair:

$$q = [s, v]$$
$$q = [s, \lambda\hat{v}]$$
$$|q| = \sqrt{(s^2 + \lambda^2)}$$

where $\hat{v}$ is the unit norm of v and λ is $||v||$, the modulus of v.

It can be derived that:

$$qq^* = |q|^2$$

Example:

$$q = 1 + 2i + 2j + 4k$$
$$|q| = \sqrt{(1^2 + 2^2 + 2^2 + 4^2)}$$
$$|q| = 5$$

A quaternion with a unit norm is called a normalised quaternion or unit-norm quaternion, represented by *q'*.

To normalise a quaternion, *q*, it is divided by its norm or magnitude, |*q*|:

$$q' = \frac{q}{|q|}$$

2.5. Multiplication of quaternions

To multiply quaternions, the same multiplication rules for imaginary operators are used as other numbers, i.e., all the terms are multiplied out and terms are simplified with the rules concerning *i, j, k*:

$$q_a q_b = [\, s_a\,,\ v_a\,]\,[\, s_b\,,\ v_b\,]$$
$$= [s_a s_b - v_a \cdot v_b\,,\ s_a v_b + s_b v_a + v_a \times v_b]$$

Matrices can be used to express a quaternion product, by converting the quaternions into matrices and working out the product of a 4x4 matrix and a column vector [4]:

$$\begin{aligned}[s_a,\ \mathbf{a}]\,[s_b,\ \mathbf{b}] = [&s_a s_b - x_a x_b - y_a y_b - z_a z_b,\\ &+ s_a\,(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) + s_b(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k})\\ &+ (y_a z_b - y_b z_a)\mathbf{i} + (z_a x_b - z_b x_a)\mathbf{j} + (x_a y_b - x_b y_a)\mathbf{k}]\\ = &\begin{bmatrix} s_a & -x_a & -y_a & -z_a \\ x_a & s_a & -z_a & y_a \\ y_a & z_a & s_a & -x_a \\ z_a & -y_a & x_a & s_a \end{bmatrix} \begin{bmatrix} s_b \\ x_b \\ y_b \\ z_b \end{bmatrix}.\end{aligned}$$

Here is an example:

$$q_a = [1 - 2i + 3j - 4k]$$
$$q_b = [5 + 6i - 7j + 8k]$$

$$q_a q_b = 70 -8i + 0j -16k$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ -2 & 1 & 4 & 3 \\ 3 & -4 & 1 & -2 \\ -4 & -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -7 \\ 8 \end{bmatrix} = \begin{bmatrix} 70 \\ -8 \\ 0 \\ -16 \end{bmatrix}$$

2.6. Division of quaternions

The division of quaternion is achieved by multiplying the inverse of the quaternion divisor. A quaternion multiplied by its inverse is always equal to 1.

The following formula can be used to work out the inverse of a quaternion, *q*:

$$q^{-1} = \frac{q*}{|q|^2}$$

where q^* is the complex conjugate of *q* and $|q|^2$ is the square of the norm of *q*.

3. 3D Rotation Transforms in Computer Graphics

In computer graphics, 3D rotation is the process of rotating an object with respect to an angle in a three-dimensional space. Using Euler Angles may feel like a more intuitive way of representing 3D rotation, however, most software developers use quaternions as they are more efficient to calculate and less problematic than Euler Angles. In this Section, both methods for calculating 3D rotations are discussed together with their efficacy.

3.1. Euler Angles for 3D Rotations

The traditional method for rotating points is based upon Euler angle rotations. A rotation in 3 dimensions consists of 3 separate rotations about each one of the Cartesian axes. This can be represented as a rotational matrix [8].

Once the point *P*(*x, y, z*) has been rotated in all three dimensions, it is notated as *P*(*x', y', z'*).

Here is the rotational matrix required to rotate the point *P*(*x, y, z*) through an angle γ about the z-axis, where the z-coordinate remains untouched:

$$\begin{bmatrix} x' & y' & z \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Geometrically, this rotation about the z-axis would look like:

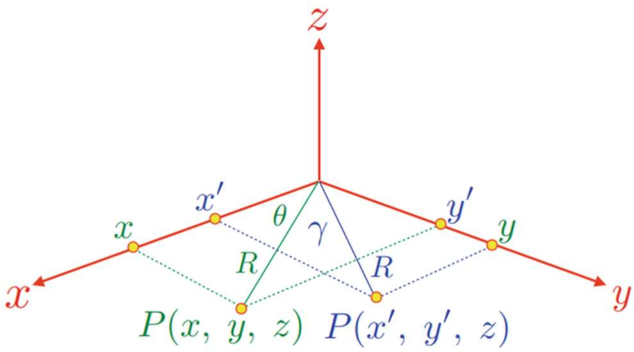

Figure 2 – Rotation about the z-axis

where θ represents the angle that *P* makes with the x-axis and γ represents that angle which P is to be rotated through.

To rotate in 3 dimensions, point *P* also rotates about the x and y axis. These can be calculated using the following matrix products, similar to the product used for rotation about the z-axis.

Rotation about the x-axis through an angle α:

$$P\left(x,\ y',\ z'\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation about the y-axis through an angle β:

$$P(x',\ y,\ z') = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To summarise, 3 individual rotations are applied for the Euler rotation method, one about each Cartesian axis.

3.2. Problems with using Euler Angles for Computer Graphics

1) When two or more rotations are combined, it is difficult to visualise and predict how the final rotation behaves

2) There are 12 different possible Euler angle rotation sequences - XYZ, ZYX, XZY, etc. To derive a set of Euler angles, a particular rotation sequence must be used. However, the routine that builds that matrix has to make some assumption about the order of

those three rotations. This could be a problem as matrix multiplication is mostly non-commutative.

3) Gimbal lock may occur, in which two of the three gimbals are driven into a parallel configuration. This results in a loss of access to one of the object's rotational axes, degenerating the three-dimensional rotation into a two-dimensional rotation [12].

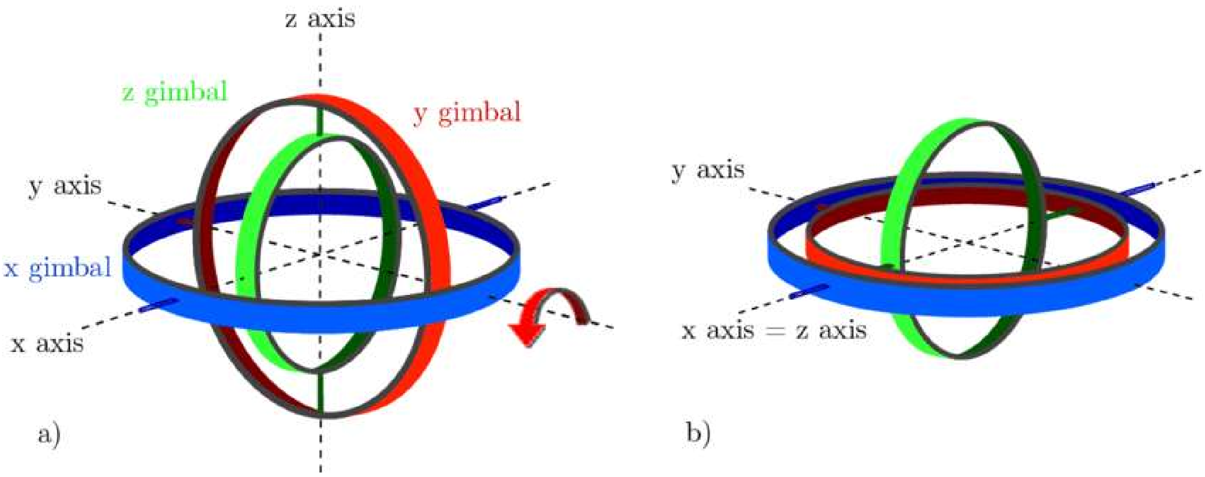

Figure 3 – Gimbal lock

The figure above illustrates gimbal lock. The blue gimbal represents the x axis, the red gimbal the y axis and the green gimbal the z axis. After a rotation of 90 degrees about the y-axis, the blue and green gimbal possess the same rotation axis. This leads to the loss of a rotational axis.

3.3. Quaternions for 3D Rotations

A useful application of quaternions is to interpolate the orientations of key-frames in computer graphics.

Hamilton had hoped that a quaternion could be used as a complex rotor [9]. It was well known that 2 dimensional complex numbers can be used as complex rotors. For example, the following complex number rotates another complex number through θ:

$$\mathbf{R}_\theta = \cos\theta + i\sin\theta$$

A unit norm quaternion, *q*, can be used to rotate a vector, which is a pure quaternion *p*, where *q* and *p* are defined as follows:

$$q = [s,\ \lambda\hat{\mathbf{v}}], \quad s, \lambda \in \mathbb{R}, \quad \hat{\mathbf{v}} \in \mathbb{R}^3,$$

$$\|\hat{\mathbf{v}}\| = 1$$

$$s^2 + \lambda^2 = 1$$

$$p = [0,\ \mathbf{p}], \quad \mathbf{p} \in \mathbb{R}^3.$$

The Sandwich Product

In order to see how the vector *p* moves when rotated by a quaternion, the sandwich product is used as follows:

$$p' = q \times p \times q^{-1}$$

The vector *p* is pre-multiplied by *q* and post-multiplied by the inverse of *q*. The sandwich product *p'* gives the new position of *p* after rotation [7].

This formula comes from the isomorphism between the quaternion group and the position of a point on a rotating object.

As an example, a vector *p* is rotated 45° about the z-axis:

$$\begin{aligned} \hat{\mathbf{v}} &= \tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k} \\ \mathbf{p} &= 2\mathbf{i} \\ q &= [\cos\theta, \sin\theta\hat{\mathbf{v}}] \\ p &= [0, \mathbf{p}] \end{aligned}$$

The sandwich product can be used to find the value of the vector *p* after the transformation:

$$\begin{aligned} q &= \left[\cos\theta, \sin\theta\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\right] \\ q^{-1} &= \left[\cos\theta, -\sin\theta\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\right] \end{aligned}$$

$$\begin{aligned} qp &= \left[-1, \sqrt{2}\mathbf{i} + \mathbf{j}\right] \\ qpq^{-1} &= \left[-1, \sqrt{2}\mathbf{i} + \mathbf{j}\right] \tfrac{1}{2}\left[\sqrt{2}, -\mathbf{i} - \mathbf{k}\right] \\ &= \tfrac{1}{2}\left[-\sqrt{2} - \left(\sqrt{2}\mathbf{i} + \mathbf{j}\right) \cdot (-\mathbf{i} - \mathbf{k}), \mathbf{i} + \mathbf{k} + \sqrt{2}\left(\sqrt{2}\mathbf{i} + \mathbf{j}\right) - \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}\right] \\ &= \tfrac{1}{2}\left[-\sqrt{2} + \sqrt{2}, \mathbf{i} + \mathbf{k} + 2\mathbf{i} + \sqrt{2}\mathbf{j} - \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}\right] \\ &= \left[0, \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}\right] \end{aligned}$$

This gives a new pure quaternion and the norm of the vector (its absolute distance from the origin) is maintained.

The figure below shows the result of the transformation:

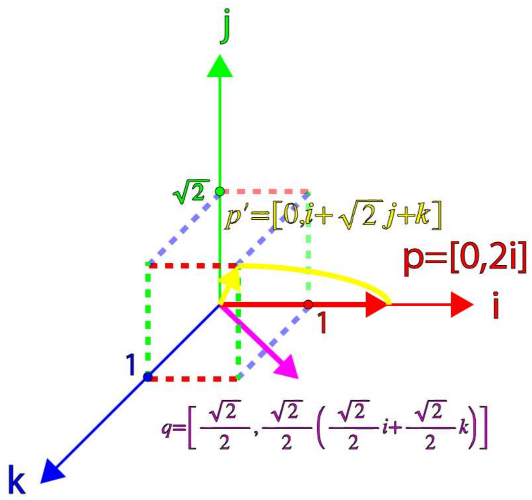

Figure 4 – Rotation of p about the z-axis

3.4. Advantages of using Quaternions for Computer Graphics

1) Representations of 3D rotations using 3 scalar values are non-linear, they have singularities and are difficult to combine. To avoid this problem, this space is imbedded into a higher dimensional 4D space as a quaternion, instead of directly modelling the 3D space.

2) Rotations defined by quaternions can be computed more efficiently and with more stability. Quaternions are represented as a four element column vector, (w,x,y,z), whilst Euler angle rotations have a 3x3 matrix representation. This means that computing quaternion rotations requires fewer computing overheads and less memory, making them more efficient [11]. This is especially important for processing graphics in modern computer games as efficiency is vital for the game to run smoothly due to the high processing requirements.

3) Unit quaternions do not have the problem of gimbal lock. This is because the unit quaternions naturally form a 3-sphere, so there are no topology issues [10]. In computer graphics, this means that rotations can be performed smoothly in all 3 dimensions, without the loss of any one dimension due to gimbal lock.

3.5. Conclusion

Quaternions are a far superior way of representing 3D rotations in computer graphics as it avoids problems with Euler angles, such as gimbal lock and computational overheads. For this reason, quaternions are heavily used in computer graphics applications such as video games. Quaternions also have vital applications in the real world. They are used for aerospace control systems for aircraft and rockets, due to the high risk of gimbal lock.

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