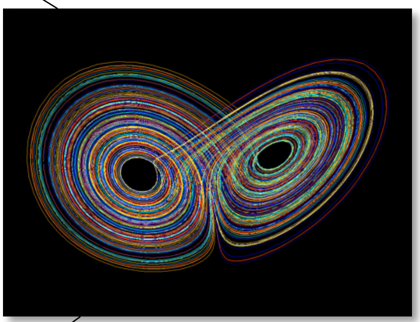
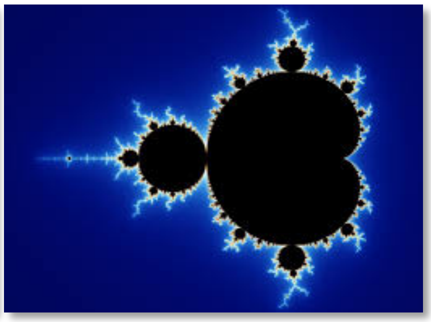
The Buzz and Sting of Chaos

Which properties do the images below all have in common?

Unpredictability

Recursion





Self-Similarity

Determinism

It turns out, these properties are fundamental to a rapidly emerging area of Mathematics I have chosen to explore.

In the 1950s, whilst experimenting with non-linear models of weather prediction, Edward Lorenz made a discovery which has completely revolutionised modern Mathematics. Initially, the results of his forecasting models were incredibly encouraging: they mirrored real life weather patterns. This led him to explore the system further by running the program for a longer time period. This time the computer inputs were rounded to 3 decimal places (instead of 6 decimal places). At first glance the results were comparable, however as time progressed, the figures diverged quicker and quicker. How could a 1/10,000 change in initial conditions lead to such a drastically different outcome?

This was the beginnings of the intriguing concept known as Chaos Theory which led to the belief that even our most powerful computers could not produce accurate long-term predictions. For this reason, it is known as “the science of the unpredictable”. Is this inability of prediction due to having inaccurate input information or will our methods of prediction always be imperfect?

To explore the idea of Chaos, imagine we are studying the population of bees in a flower garden. Let’s say that the theoretical maximum population size of bees in the garden is 100 as there is a restricted amount of pollen available in the garden to maintain the bee population.

The question is:
“How can we predict how many bees will be left in the garden next month?”

Well… we can use a logistic map to represent this. A logistic map is a one-dimensional discrete-time map which shows how chaotic, unpredictable outcomes can arise from a relatively simple recursive mathematical equation. A logistic map is intended to show how population varies over time (n) while also representing the phenomenon of Mathematical Chaos.

A logistic map relies on 2 variables: the first of which is the initial population (as a value between 0 and 1) which is represented as x. If the maximum population is 100, then when…

x=1 ⇒ P= 100 bees
x=0 ⇒ P= 0 bees
x=0.65 ⇒ P= 65 bees

The map also relies on r, which is a constant value representing the population growth over time. The value of r is crucial in the formula and takes into consideration factors such a birth rate, death rate, competing insects, weather conditions and human behaviour. I won't explain how to calculate "r" however, figures 1-4 show how small changes in the r rate do affect the outcomes.

The overall equation for our logistic map is therefore:

$$x_{n+1}=rx_n(1-x_n)$$

For example, if the initial population of bees= 65 ⇒ x_0= 0.65 and for this trial, I am going to set r=3.9. If I substitute these values into the formula I get:

x_1= 3.9*0.65(1-0.65)
x_1= 0.88725

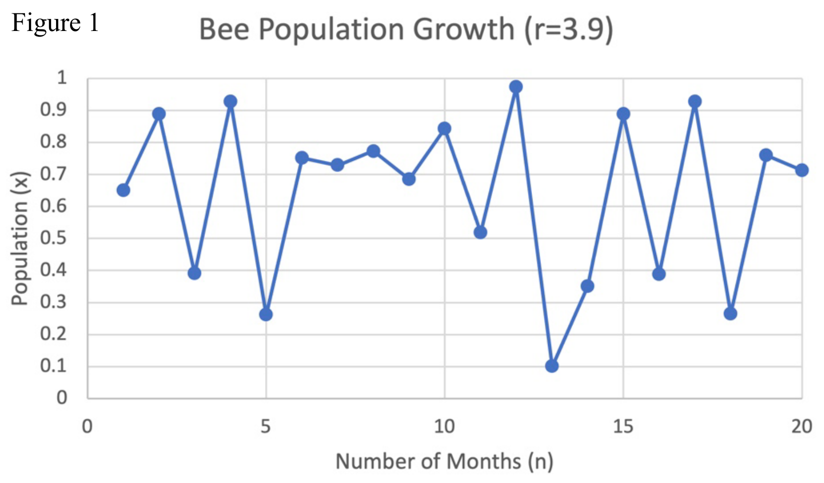
This means that after one month the population will increase to 89- these are some busy bees we have here! What do you now expect to happen to the population one month later?

Well… if we now say that x_1= 0.88725 and r=3.9 (as it is a constant):

x_2= 3.9*0.88725(1-0.88725)
x_2= 0.39014601.

This means that the population has now dramatically decreased one month later down to 39. This for example may be because there are a limited number of flowers in the garden which can only feed a certain number of bees at a given time.

Figure 1



On the scatter diagram above, the number of months, n, is plotted against population, x_n

Figure 1 shows how the Population varies over just 20 months; however, this recursive relation can continue and for example the population of bees at 1000 months would be 11. This diagram is a perfect example of Mathematical Chaos (behaviour that appears random but is not actually random). Without a given formula this behaviour seems almost impossible to predict, but in theory the system has a fully **deterministic** future if we have precise information about the initial conditions. In practice however, accurate information is difficult to obtain for a large population or a population that fluctuates regularly. This means that determinism doesn't directly imply predictability, which is at the heart of Chaos Theory.

The next important question is what happens when the value of "r" changes?

Figure 2

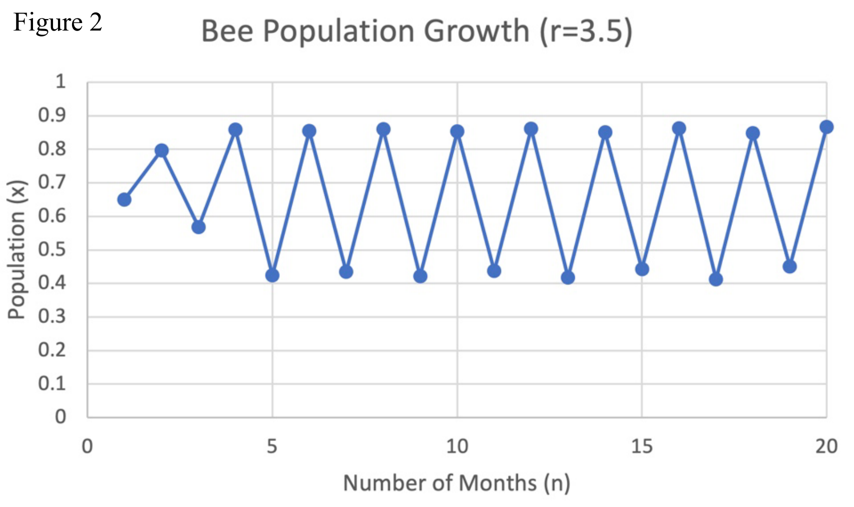


Figure 2 shown above has a period of 2 and is oscillating between 0.4 and 0.86 once it has stabilised. What I think is fascinating to note is that a seemingly small difference in the r rate of 0.4 has changed the nature of the system from being completely unpredictable to a relatively repetitive pattern. This value for 'r' represents the **recurrence** involved in Chaos theory. There is potential for a repeating unit greater than 2 at different values of r, you could even have outcomes with a repeating unit of length 100!

The stark contrast between the Figure 1 and Figure 2 also demonstrates the sensitivity on the initial conditions. This sensitivity on initial conditions is popularly known as the Butterfly Effect. The Butterfly Effect is the notion that the flap of a Butterfly's wings in Brazil can result in a tornado is Texas. In everyday life, the Butterfly effect is seen as a metaphor to represent that small activity matters (even the buzzing of bees) and shows us that all living creatures are part of a wider interconnected universe. In recent years the Butterfly effect has become an incredibly prominent in the media meaning people uphold a misconception of its purpose. The media captures the Butterfly effect as a negative warning that tiny changes in life may result in enormous repercussions. Contrastingly, the focus in Mathematics is that initial conditions may be so sensitive that no matter our accuracy, the calculated results may still differ hugely. Using this idea; if a Butterfly's wing flap can result in a tornado, equally can a bee buzzing stop it?

Figure 3

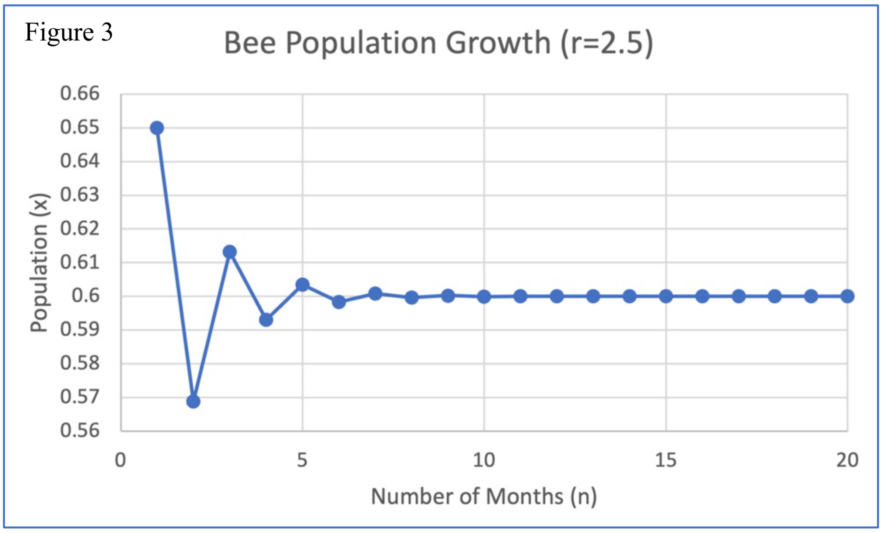


Figure 3 above shows what happens when r=2.5. Here, initially the population increases and decreases chaotically however after approximately 6 months the population stabilises in size at an equilibrium of x=0.6 (so P= 60 bees) which is quite common behaviour in many animal kingdoms. For this value of r, no matter the value of the initial population, the number of bees will always stabilise at 60.

Figure 4

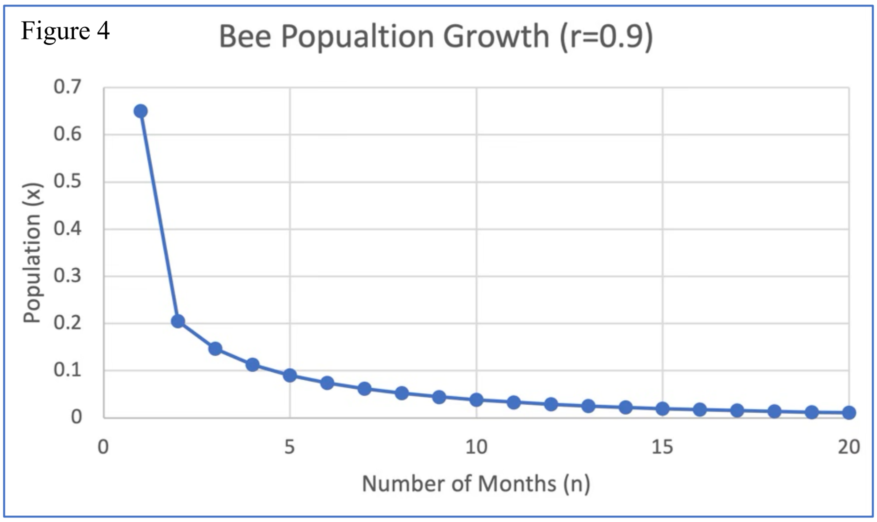


Figure 4 demonstrates that if a bee population has an r rate of < 1 it will become extinct. This trend can be comparable to r=2.5 (figure 3) as both models show the population tending to a fixed equilibrium population whether that be 0 or 60!

Having explored the outcomes arising from a range of different r values, it poses the question: is there a value for r which is in a sense a ‘tipping point’ for chaos? This can be visualised when the r rate is plotted against the equilibrium population, (see figure 5 below).

Figure 5

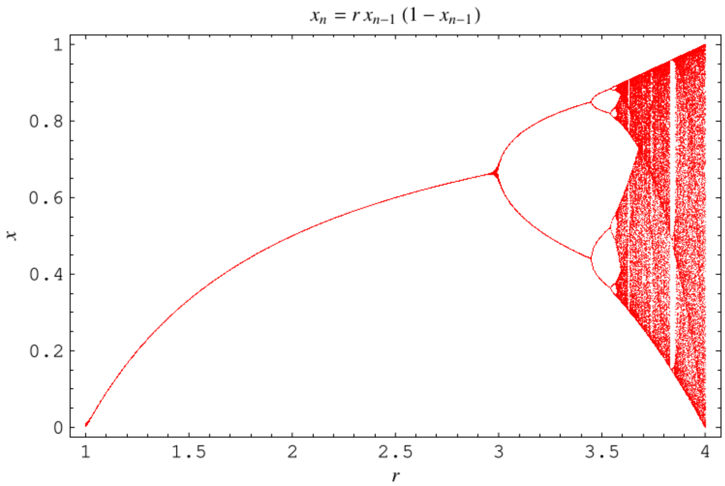
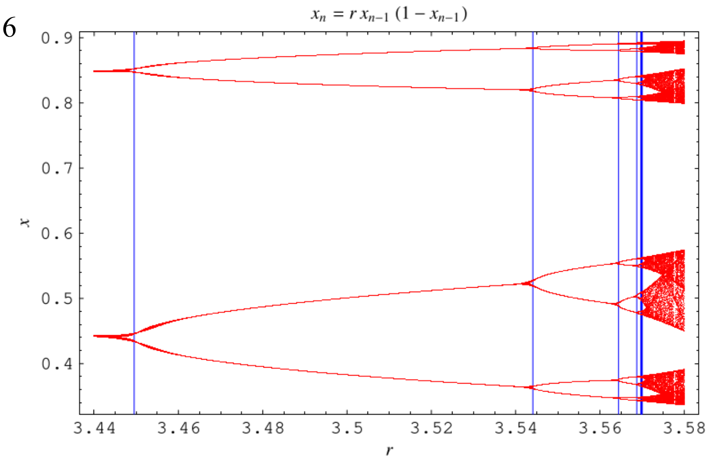


Figure 5 shows several fascinating patterns as noted below:

1. For values of r<1, equilibrium population remains at zero as demonstrated in Figure 4.
2. When r>1, the equilibrium population stabilises at a constant value greater than zero which increases in magnitude as r increases. For example, in figure 3 when r=2.5 an equilibrium population of 0.6 (60) is reached.
3. When r=3, the graph appears to split in 2. This shows that the equilibrium population is oscillating between 2 values (as shown in figure 2 when r=3.5 and the logistic map oscillated between 0.4 and 0.86). This cyclical growth pattern is realistic in many animal kingdoms.
4. As r continues to increase, the cycle length of equilibrium populations doubles to 4 then 8 etc which is shown by the continual splitting of the lines. So, to answer my question above, the tipping point for Chaos is reached approximately when r=3.56995 which can be observed in Figure 5 where the bifurcation pattern appears to spiral out of control. However, there are other stable values beyond r=3.56995 such as r=3.83 which has an equilibrium population cycle with a period of 3. This figure is in a sense the ‘eye’ amongst ‘the tornado of chaos’.
5. The behaviour portrayed in figure 5 is known as **Bifurcation.** The bifurcation diagram is in fact a type of fractal.

Fractals are infinitely complex patterns that are self-similar across different scales. Driven by recursion, fractals are images of dynamical systems- the pictures of Chaos. If I zoom in on the bifurcation diagram in the domain of 3.44<r<3.58, fascinatingly many of the same images continue reappearing! This property is known as self-similarity (as shown in figure 6 below)

Figure 6



Not only do fractals relate to the population growth of bees, but they also relate to the habitat in which they live in: the natural world. Many flowers also exhibit infinitely complex patterns revealing the chaos amongst the perfection and beauty of nature.

The Queen Anne's Lace herb (pictured to the left) displays fractal properties as the whole plant looks almost identical to a close up image of a singular branch suggesting it is a self-similar plant.

Refocusing on the bifurcation diagram, I think this way of representing the relationship between r and initial population reiterates that population growth is so incredibly sensitive on initial conditions; for example, r=3.83 will create an equilibrium population with a period of 3 however, r=3.9 (Figure 1) portrays pure chaos.

This is a problem in modelling as Mathematicians will have to use some degree of approximation to calculate a value for "r" and the initial population. I think the quote below perfectly encapsulates the Chaos associated with predicting population growth:

"Chaos: when the present determines the future, but the approximate present does not approximately determine the future" **Edward Lorenz**

To conclude, I think that Chaos theory provides unlimited possibilities in fields such as Population, Astronomy, Meteorology and even Economics. However, chaos does limit our ability to make predictions for non-linear systems. Can we ever obtain complete and accurate information on the sensitive initial conditions?

Are our lives the perfect example of chaos? A unique, deterministic path that is completely unpredictable.

Sources:

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