The Painter's Paradox

Suppose a person came up to you and said that he bought a water bottle that can be filled up completely with 500 grams of water, but if you want to wet the surface of the water bottle, you will need at least 500 tons of water. What would you think? Would you believe that it is true or not?

In mathematics, there isn't such a bottle, but there is a problem exactly like it, which is known as the 'Torricelli's Trumpet' problem, or the 'Gabriel's Horn' as some call it. It has been named as the shape of the object looks really similar to a horn as shown in Figure 1. For ease of remembrance, it also has another name called the 'Painter's Paradox'.

Similar to the water bottle example, the weird thing about this horn is – firstly, it is infinitely long and secondly, if you want to fill it up with paint, you can do that, but if you want to paint the whole surface, then very sorry, it is not manageable. Because theoretically, you will need an infinite amount of paint to cover it all up.

The properties of this particular geometric figure, finite volume with an infinite surface area, were first studied by an Italian physicist and mathematician, and a student of Galileo, Evangelista Torricelli (as shown in Figure 2) in the early 17th century.

Figure 1 Torricelli's Trumpet

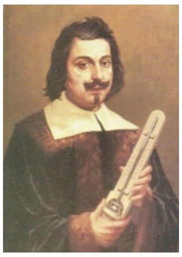

Figure 2 Portrait of Evangelista Torricelli

We start off by looking at the graph of $y=\frac{1}{x}$ ($x\geq 0$) as shown in Figure 3 (a). When x approaches to zero, y tends towards infinity, but as x gets bigger and bigger, y decreases and tends towards 0. However, for the Gabriel's Horn, we only consider the graph which $x\geq 1$ and gives that $y\leq 1$ as shown in Figure 3(b).

In order to make a Gabriel's Horn, we will have to rotate the graph around the x-axis in three dimensional space and thus create a shape like the horn. Hence why the word 'Horn' is included in the name. The conclusion was very surprising at that time and even now, because intuitively speaking, it was inconceivable that an object with a finite volume could have an infinite surface area since the shape goes on infinitely so surely the volume would be infinite as well, since there should be a little bit more paint that you can pour in every time.

Yes, because this little horn was so counter-intuitive that the famous British philosopher of the time, Thomas Hobbes claimed, 'If you want to understand it from the senses, only a mad man can do it!' Hence this is also the reason why it is called a paradox.

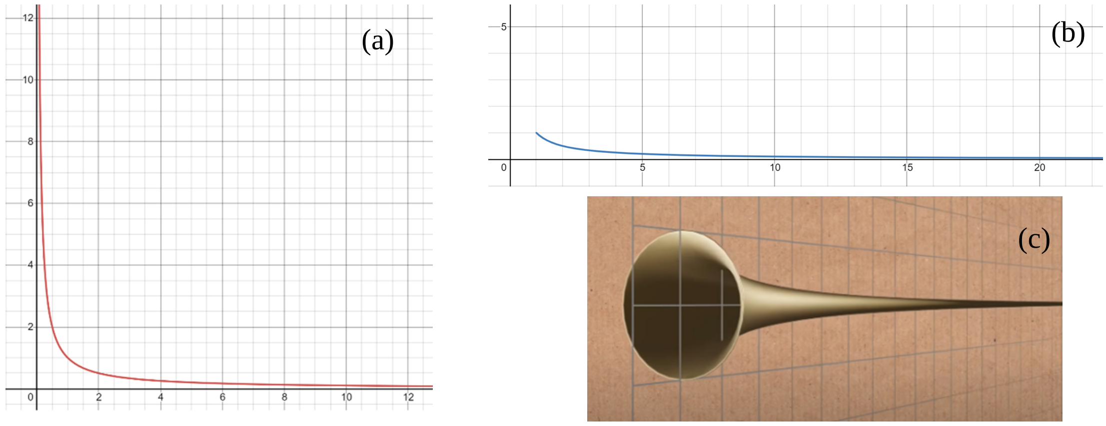

Figure 3 Generation of the Gabriel's Horn

Instead of focusing on the intuitive side, we will try and prove it using maths. When the discovery was made, it used Cavalieri's principle, which was before the invention of calculus and it is the early step towards integral calculus. Cavalieri's principle, named after Bonaventura Cavalieri, is a modern implementation of the method of indivisibles which basically is a theorem that conveys that if two figures have the same height and the same cross-sectional area at every point along that height, they have the same volume. As show in Figure 4, the two stacks of coins with the same cross section areas should have the identical volumes based on the Cavalieri's principle.

Figure 4 Two stacks of coins with the same volume

To find the volume, we will need to cut the object into a lot of really small slices vertically i.e. parallel to the y-axis and find the sum of all the slices' volumes as shown in Figure 5(a). As the slices gets smaller and smaller, it is basically doing integration which has a definition of a way of adding slices to find the whole. Based on the shape of the $y=\frac{1}{x}$ graph, the slices are conical frustums and look similar to a cylinder but not perfect cylinder as the edge slightly slanted (Figure 5 (b)). However, if you make each slice small enough, the slant will become negligible small and you can assume that volume of that slice equals to the volume of the cylinder, which is $\pi r^2 h$, where r is the radius of the cylinder and h the height of the cylinder.

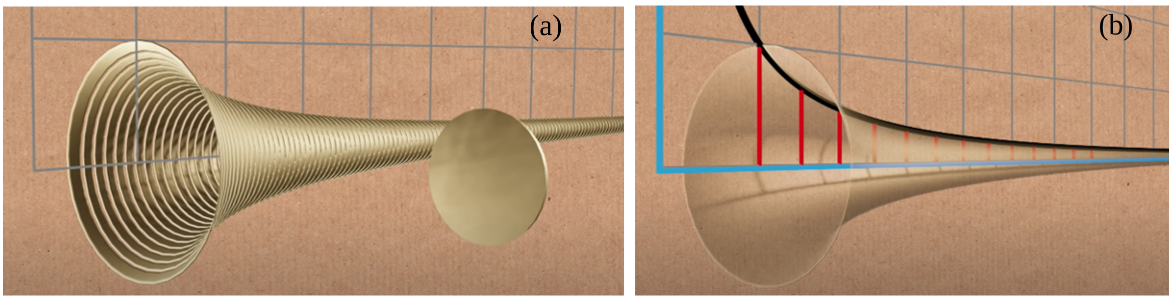

Figure 5 Segmentation of the Gabriel's Horn

The height of the cylinder will therefore be the change in x value denoted dx. So the volume of each slice will be $\pi r^2\, dx$. In order to find the total volume of the horn, we need to add up all the values for each slice which gives us the following integral:

$$V = \int_1^{\infty} \pi r^2\, dx \tag{1}$$

We will consider one slice of the object to find the radius of it. As it is a body of revolution around the x-axis, so the points on the x-axis will be the centre of the circle and the distance from the graph to the x-axis which is the corresponding y values will be the radius of the cylinder. Therefore, we get:

$$r = y = \frac{1}{x} \tag{2}$$

Substituting Equation (1) into Equation (2), yields:

$$V = \int_1^{\infty} \frac{\pi}{x^2} dx = \pi \int_1^{\infty} \frac{1}{x^2} dx = \pi \left[-\frac{1}{x}\right]_1^{\infty} \tag{3}$$

As $x \to \infty$, $-\frac{1}{x} \to 0$, Equation (3) becomes:

$$V = \int_1^{\infty} \frac{\pi}{x^2} dx = \pi \left[-\frac{1}{x}\right]_1^{\infty} = \pi\,[0-(-1)] = \pi \tag{4}$$

This proves that the volume of paint needed is finite - π units, which means the horn will be able to be filled up with paint at some point.

Now, let's consider the outside surface area around the horn. If we still think about the horn as slices and the total surface area will be the sum of each individual surface area of each slice, so take one slice as consideration. Cut it open along a line which gives a shape of an annular sector as shown in Figure 6, the area of this shape, gives the lateral surface area of the corresponding conical frustum, i.e. the slice of the object.

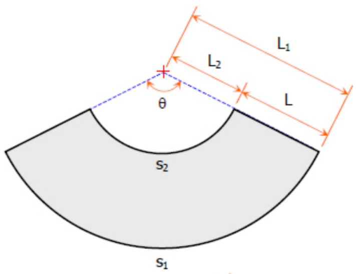

Figure 6 Expansion of the lateral surface of conical frustum

Let the annular sector be the area difference of two sectors with arc length S_1 and S_2, where $S_1 > S_2$ with common central angle θ, and the radii of circles denoted L_1, L_2 and the distance of the slant is L.

Using the equation for area of a sector $A = \frac{1}{2}\theta r^2$, the Area of the annular sector = area of sector1 - area of sector 2$= \frac{1}{2}\theta L_1^2 - \frac{1}{2}\theta L_2^2 = \frac{1}{2}\theta(L_1 + L_2)(L_1 - L_2) = \frac{1}{2}\theta(L_1 + L_2)\cdot L = \frac{1}{2}(S_1 + S_2)\cdot L$.

As the slices number gets more and more, the slants become shorter and shorter, the difference between S_1 and S_2 gets smaller and smaller so we can assume that $S_1 = S_2 = S$, therefore, $\frac{1}{2}(S_1 + S_2)\cdot L \approx SL$

In order to find the total surface area of the horn, we will need to add up all the lateral areas of the conical frustum, which forms this integral:

$$\text{Total surface area} = \int_1^\infty S\,dL = \int_1^\infty 2\pi \frac{1}{x} dL \qquad (5)$$

where dL is the distance of the slant and x is the point which the radius is taken. As dL gets smaller and smaller, the curve would eventually look really similar to a straight line so that could find an expression of the value of dL as if it is a straight line. Consider a right angle triangle formed under dL with dx and dy. Using Pythagoras theorem, we will be able to form the equality $dL^2 = dx^2 + dy^2$, therefore:

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{1 + \frac{dy^2}{dx^2}} \cdot dx = \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \cdot dx \qquad (6)$$

Substituting Equation (6) into Equation (5), yields:

$$\text{Total surface area} = \ 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx \qquad (7)$$

Since $x \in [1, \infty)$, thus $0 < \frac{1}{x^4} \leq 1$, therefore $1 < \sqrt{1 + \frac{1}{x^4}} \leq \sqrt{2}$, we have

$$\text{Total surface area} > 2\pi \int_1^\infty \frac{1}{x} dx = 2\pi [Lnx]_1^\infty \to \infty \qquad (8)$$

As the total surface area is greater than something that goes to infinity, so it is valid to conclude that the total surface area also goes to infinity. As a result, we have mathematically proven by integration that it is able to fill up the horn with a finite amount of paint and unable to cover the whole surface of the horn which forms the Painter's Paradox.

The surprising conclusion is formed base on the fact that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, i.e. the value of the sum never becomes closer and closer to a constant, they continue to increase or decrease and approach to infinity as n approaches infinity whereas $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, i.e. you achieve closer and closer to a constant value that is not infinity as n approaches infinity.

There is a simple way to think about why the sum of $\frac{1}{n}$ diverges as:

The 1st term: $1 > \frac{1}{2}$;

The 2nd term: $\frac{1}{2} = \frac{1}{2}$;

The 3rd + 4th term: $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$;

The next 4 terms: $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

The next 8 terms $> \frac{1}{2}$

The next 2^x terms $> \frac{1}{2}$

…

As the sequence is never ending, there will always be some terms added on that is greater than $\frac{1}{2}$, so that we can always add $\frac{1}{2}$ to the sum, and if the sum of the sequence is bigger than something that is infinite, the sum of the sequence itself is infinite, which means as $n \to \infty$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

To prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we could use a similar method and prove that it would always be bigger and smaller than a value that is not infinity which means it will never get to infinity.

since $\frac{1}{n^2} < \frac{1}{n(n-1)}$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n(n-1)}. \tag{9}$$

Using method of differences,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n-1)} &= 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{\mathrm{n}-1} - \frac{1}{\mathrm{n}} \\ &= 2 - \frac{1}{n} < 2 \end{aligned} \tag{10}$$

Since n are positive integers, and thus the sum should be larger than zero, yields,

$$0 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2 \tag{11}$$

Accordingly, as $n \to \infty$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

A more precise value ($\frac{\pi^2}{6}$) of this series converge into could be proven using the integral test and many other mathematical method, but knowing that it converges is enough to explain why it is finite.

Since the volume is based on the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and the surface area is based on $\sum_{n=1}^{\infty} \frac{1}{n}$ which explains why the volume is finite but the surface area is infinite.

The maths is correct but the logic seems wrong to people, how could you have an object that you fill with finite amount of paint but needs infinite amount of paint to cover the surface of the object in real life. So, what is the problem here?

The problem lies in the human understanding and application of the concept of infinity. In fact, more than 2000 years ago, Aristotle, the great scholar of ancient Greece, recognised that in mathematics, one can only accept potential infinity and not actual infinity, otherwise one will get absurd results.

Potential infinity is a group of 'things' e.g. numbers that continues without ending, it continuously goes on or repeats itself without an ending point that is recognizable. An example of potential infinity is the group of natural numbers, the next element in the group is always the previous add 1 and there is no particular number where the sequence will end.

Actual infinity is the infinite partitioning of an already formed, real, static object of a certain size and shape, which is always inexhaustible. For example, dividing a finite length of stick into infinite number of pieces. However, modern science has proven that our universe is a quantum universe and that the smallest unit of matter is quarks so when the stick is divided to the scale of a quark, it cannot be divided any further as quarks are indivisible.

Therefore, according to Aristotle’s theory, a horn is an object that has already been formed as it has been manufactured, has a fixed weight, length so as a consequence, infinity cannot apply to it. So if we want to find the surface area and its volume, a necessary condition is that it’s already made and has a definite and invariable shape. But in the previous calculations, we did the calculation based on that the length of the trumpet is infinite which implicitly admitted the existence of actual infinity, which is equivalent to the trumpet being always lengthening and the growth in size never ends which will not exist in real life.

Behind this problem, it is the two different perspective of the views of truth: idealism, which believes in that mathematical proof is the ultimate prove of truth and materialism, which insists that social practice is the ultimate prove of truth, hence why this is called a paradox.

References:

All images used are either my own drawings using Desmos or general pictures from Internet.