SOLUTIONS OF THE DIOPHANTINE EQUATION $x^3+y^3+z^3=k$

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1. In a recent paper, Mordell† discusses solutions of the equation

$$x^3+y^3+z^3+w^3=m \tag{1}$$

and gives an outline of available information on this equation and others derived from it. In particular he remarks " I do not know anything about the integer solutions of

$$x^3+y^3+z^3=3$$

beyond the existence of the four sets (1, 1, 1), (4, 4, −5), etc.; and it must be very difficult indeed to find out anything about any other solutions ".

At Prof. Mordell's suggestion, an attempt was made to find further solutions with the help of the electronic computer (EDSAC) at the Cambridge University Mathematical Laboratory. This paper records the results obtained during that search, which was extended to include all solutions of

$$x^3+y^3+z^3=k \tag{2}$$

with $0 \leqslant k \leqslant 100$, $|z| \leqslant |y| \leqslant |x| \leqslant 3164$. No further case with $k=3$ was found; 345 primitive solutions, for which $(x, y, z)=1$, $|x| \leqslant 3164$ are listed below. There are also 91 derived solutions, for which $(x, y, z) > 1$.

2. It is a reasonably simple task to program a direct trial and error search for solutions, and two such programs were prepared, one by each author. For two main reasons, the search was not confined to the case $k=3$: (i) it is clear that solutions for other values of k also have their interest, (ii) it is desirable to present a problem to the machine in such a way that solutions are produced at reasonably frequent intervals. It is also relevant to note that it takes very little longer to search for solutions with a wide range of k than to search for solutions with a single value of k.

The second point is important for the operator, since the production of an occasional solution provides a useful indication that the machine is working correctly, and at the same time tends to preserve an interest

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† L. J. Mordell, " On the integer solutions of the equation $x^2+y^2+z^2+2xyz=n$ ", *Journal London Math. Soc.*, 28 (1953), 500–510.

in the computation. The limit $k \leqslant 100$ was chosen rather arbitrarily, but proved satisfactory.

3. The search is at first sight " three-dimensional ", that is, x, y, z may apparently vary independently over considerable ranges of values. However, the inequality $x^3+y^3+z^3 \leqslant 100$ is, in fact, sufficiently restrictive to ensure, for by far the greater part of the search, that at most two values of y are tested for each combination (x, z). The search is thus " two-dimensional " in practice.

We must now specify the problem a little more closely. Clearly we may, without loss of generality, take

$$|x| \geqslant |y| \geqslant |z|, \tag{3}$$

and by changing all signs if necessary, and allowing k to lie anywhere in the range

$$-100 \leqslant k \leqslant 100, \tag{4}$$

we may suppose that $x > 0$. This will be assumed throughout this and the following paragraphs 3 to 8, although in the tables we still keep $k > 0$ and allow x to be negative.

Four sign combinations are now possible

$$(x, y, z) = (l, m, n) \tag{5.1}$$

$$(x, y, z) = (l, m, -n) \tag{5.2}$$

$$(x, y, z) = (l, -m, n) \tag{5.3}$$

$$(x, y, z) = (l, -m, -n) \tag{5.4}$$

in which $l \geqslant m \geqslant n \geqslant 0$. We consider these cases in turn.

4. Case (5.1). Since x, y, z are all positive, while $l^3+m^3+n^3 \leqslant 100$, it is clear that $l \leqslant 4$, and that there is a small number of solutions which were obtained by inspection.

Case (5.2). Here $100 \geqslant x^3+y^3+z^3 = l^3+m^3-n^3 \geqslant l^3$, since $m \geqslant n$. Again $l \leqslant 4$, leading to a small number of solutions, found by inspection.

Case (5.3). Here, with $100 \geqslant k = l^3-m^3+n^3$, there are the infinite sets of solutions $(l, -l, n)$ for each $0 \leqslant n \leqslant 4$, with l taking any integer value. If we exclude these obvious sets, we may write $l \geqslant m+1$, and then

$$100 \geqslant 3l^2-3l+1+n^3$$

whence $l \leqslant 6$.

Hence, once again, we have a small number of solutions that may be found by inspection.

5. Case (5.4). There remains the final and main case

$$x^3+y^3+z^3=l^3-m^3-n^3 \text{ with } l>m\geqslant n. \tag{6}$$

We may write $l>m$ strictly, if, as before, the infinite sets $(l, -l, -n)$ are excluded; these sets are identical with those of case (5.3), apart from a change in all signs. We now examine the ranges of values that may be taken by l, m, and n.

Firstly l is unrestricted in magnitude, as will be evident from the general cases given in §13.

Secondly $m\geqslant n$, so that

$$100\geqslant l^3-2m^3,$$

or

$$m\geqslant [\tfrac{1}{2}(l^3-100)]^{1/3},$$

whence, on incorporating (6),

$$l\,.\,2^{-1/3}\leqslant m+1\leqslant l, \qquad l\geqslant 6. \tag{7}$$

Thirdly $$100\geqslant l^3-m^3-n^3\geqslant l^3-(l-1)^3-n^3$$

giving $$n^3\geqslant 3l^2-3l-99 \text{ or } n\geqslant n_0(l) \tag{8}$$

where $n_0(l)$ is a limit, positive when $l>7$ and increasing with l. Also

$$l^3-2n^3\geqslant -100,$$

or

$$n\leqslant [\tfrac{1}{2}(l^3+100)]^{1/3} \tag{9}$$

whence, finally,

$$(3l^2-3l-99)^{1/3}\leqslant n\leqslant l\,.\,2^{-1/3}+1, \qquad l\geqslant 5. \tag{10}$$

6. The method used on the machine was essentially as follows. Suppose

$$E=l^3-m^3-n^3 \tag{11}$$

is known for particular values of l, m, n, restricted by (7) and (10). For the moment l is kept fixed, and the magnitude of E tested. If E exceeds 100, we increase n by a unit; if E is less than -100, we decrease m by a unit. In either case, the test and subsequent stages are then repeated.

If $-100\leqslant E\leqslant 100$, a new solution has been found and is recorded, n is then increased, and the test for magnitude and subsequent stages repeated.

It remains to be shown that, apart from a finite number of cases with small l, two successive steps cannot both lie within the range $|E|\leqslant 100$. We note first that, if such a value, $|E|\leqslant 100$, is found, the next change is always an increase of a unit in n; this was arranged deliberately because $n\leqslant m$. The corresponding change in E is least when n is least. Hence the minimum step, for prescribed l, occurs when n is at its minimum value, given by (8). The corresponding change δE, namely $(n_0+1)^3-n_0^3$, exceeds 200 when l exceeds 12, as may be seen from the following values:

$$\left.\begin{array}{lcccc} l & 10 & 11 & 12 & 13 \\ n_0 & 6 & 7 & 7 & 8 \\ \text{Step } \delta E & 127 & 169 & 169 & 217 \end{array}\right\} \tag{12}$$

The few cases with $l \leqslant 12$, when two or more solutions can exist with the same values of both l and m, may be obtained by inspection, but were, in fact, found by one of the programs (see §7); the other program (see §8) was designed for dealing with large l and omitted the special instructions needed to cope with these solutions.

7. The two programs make use of different search plans. In one case we write $l = m+r$ and

$$\begin{aligned} E &\equiv (m+r)^3-m^3-n^3 \\ &= 3m^2r+3mr^2+r^3-n^3 \end{aligned} \tag{13}$$

which is quadratic in m, cubic in r and n.

The search was made by taking $r = 1, 2, 3, \ldots, R$ in succession; for each value of r, m started at unity, whilst n started from zero. One or other of m and n was then increased by unity, according to the sign of $E-100$, recording any solution with $|E| \leqslant 100$, this process being repeated until m reached 600. The value of r was then increased by unity, and the process repeated, m and n again starting from 1 and 0 respectively.

The value of r was increased steadily to a final value R, chosen so that all cases with $n \leqslant 600$ were covered, that is, so that

$$(601+R)^3-2(600)^3 > 100 \geqslant (600+R)^3-2(600)^3.$$

This gives $R = 155$.

The changes in E consequent on changes in m or n were effected by the recording and appropriate addition of first and second differences corresponding to unit change in m, or of first, second and third differences for unit changes in n. The former depend on r, but this remains constant for each run of 600 values of m. Differences were also used to alter the starting values of quantities depending on r, when a new cycle of 600 values of m was started.

So long as $l = m+r \leqslant 12$, each change in m was combined with a fresh start at zero for n, to pick up those of the cases mentioned at the end of §6, where an increase in m and a decrease in n were simultaneously possible. When $l > 12$, however, the value of n could no longer decrease as m increased steadily to 600.

This program gave all solutions for case (5.4) with $|k| \leqslant 100$, having m, $n \leqslant 600$. A few cases with $l > 600$ were thus included.

8. The second program also used differences for modifying E, but worked directly in terms of l, m, and n. For each value of l, m was started at $l-1$ and n started at $n_0(l-1)$, [see (8)]. The first step was to determine $n_0(l)$, noting that $n_0(l) \geqslant n_0(l-1)$. The general run then decreased m by unity or increased n by unity according as $E-100 <$ or $\geqslant 0$; this process ceased when n exceeded m, and l was then increased by unity and the cycle recommenced. Solutions with $|E| \leqslant 100$ were printed as found.

Owing to the fact that the terms in E are cubic for all of l, m, n, the innermost cycle (that which changes m or n by unity) is a little longer than in the other program. However, the complete separation of l, m, n made it a little easier to start the search at an arbitrary value of l and to stop at any convenient larger value. This program was used up to $l = 3164$.

9. The main table gives all the solutions found with $|x| \leqslant 3164$. Only primitive solutions, for which (x, y, z) have no common factor, are given. The sign of k is kept positive throughout the table, so that x may be negative.

The arrangement is to list solutions for each value of k in succession and, when k has a cubic factor, to give also the number of solutions that may be derived, up to the limit for x stated above, from solutions for earlier values of k.

The values to $|x| = 1600$ have all been obtained twice from distinct runs on the machine. Above $|x| = 1600$, only one run was made, as it seemed of more interest to spend time in obtaining new solutions, readily verified, rather than to make sure that no odd solution had been overlooked. In fact, just one case, with $k = 100$ exactly, had been omitted on the first run.

Discussion of results.

10. The total numbers of solutions found, for successive ranges of 500 in $|x|$, are listed below

Range of $\lvert x \rvert$		1-500	501-1000	1001-1500	1501-2000	2001-2500	2501-3000	Total
Solutions	Primitive	232	44	19	18	18	11	342
	Derived	49	11	9	8	7	5	89
	Total ...	281	55	28	26	25	16	431

There were also found three primitive and two derived solutions for $3000 < |x| \leqslant 3164$.

11. Since all cubes have the form 9λ or $9\lambda\pm1$, the ways in which $k=9n\pm\kappa$ can be made up depends on κ.

$k=9n\pm4$ is impossible, while the 27 combinations (α, β, γ) where $x=3\lambda_1+\alpha$, $y=3\lambda_2+\beta$, $z=3\lambda_3+\gamma$, with α, β, γ, each -1, 0, or 1, are distributed thus

Form of k	Combinations (α, β, γ)	No. of Combinations	No. of Solutions
$9n$	(0, 0, 0) (0, 1, -1) etc. ...	7	93
$9n\pm1$	(±1, 0, 0) etc., (1, 1, -1) etc., (1, -1, -1) etc. ...	12	190
$9n\pm2$	(1, 1, 0) etc., (-1, -1, 0) etc. ...	6	112
$9n\pm3$	(1, 1, 1) or (-1, -1, -1) ...	2	41
			436

The last column gives the number of solutions, among the 436 found, falling into the four categories, indicating fewer than expected for $k=9n$ and rather more for $k=9n\pm2$.

The distribution within the groups is, however, highly erratic. Further sub-division, for modulus 27 or 81, may help to account partially for this. For example, $9n$ is composed of the three cases $27m$ and $27m\pm9$. If x, y, z have the forms $9\mu_1+\alpha$, $9\mu_2+\beta$, $9\mu_3+\gamma$, denoted by (α, β, γ) then $k=27m$ is given by the forms $(\pm3, 0, 0)$, $(\pm3, \pm3, 0)$, $(\pm3, \pm3, \pm3)$ and by $(\rho, 1, -1)$, $(\rho, -2, 2)$ and $(\rho, 4, -4)$, where $\rho=0$, 3 or 6; counting permutations this gives 81 forms. Likewise $k=27m+9$ is given by the forms $(\rho, 1, 2)$, $(\rho, -2, -4)$ and $(\rho, 4, -1)$; with permutations this gives 54 forms. Thus $27m$, $27m+9$, $27m+18$ occur in proportions $3:2:2$. The table does not provide enough evidence to give a good test of this.

12. The paucity of solutions for $k=9n\pm3$ is noticeable; this covers the interesting case $k=3$ and may be made the basis of an extended and refined search in this case.

In fact, we may allow both k and x [still retaining (3)] to take either sign, but have instead the restriction that x, y, z, shall all be of the form $3\lambda+1$. A further search along these lines is planned, with extended range of k, say $|k|\leqslant 10000$, for the reasons indicated in §1, and also in order to provide material to throw light, if possible, on some of the queries of §14.

13. Two single-parameter families of solutions have been given, respectively for $k=1$ and $k=2$. If we relax the condition (3), that

$|x| \geqslant |y| \geqslant |z|$, these may be written

$$k=1 \qquad x=9t^4 \qquad y=-9t^4+3t \qquad z=-9t^3+1 \qquad (14)$$

$$k=2 \qquad x=6t^3+1 \qquad y=-6t^3+1 \qquad z=-6t^2 \qquad (15)$$

These give a solution for each integer value of t, positive or negative.

Both (14) and (15) can be generalized to two-parameter solutions, for values of k depending on the second parameter:

$$k=\lambda^{12} \qquad x=9t^4 \qquad y=-9t^4+3t\lambda^3 \qquad z=-9t^3\lambda+\lambda^4 \qquad (16)$$

$$k=2\lambda^9 \qquad x=6t^3+\lambda^3 \qquad y=-6t^3+\lambda^3 \qquad z=-6t^2\lambda \qquad (17)$$

Neither of these contributes for $|k| \leqslant 100$, although (17) with $\lambda=2$, after removing the common factor 2, contributes to $k=128$; *e.g.* (7, 1, −6) or (85, −77, −54).

Only nine of the 21 solutions found for $k=1$ are given by (14), whereas all eight solutions for $k=2$ are given by (15).

For $k=16$, however, although seven solutions found are derived from those for $k=2$ by doubling x, y, z, a single solution has been found (1626, −1609, −511), which is not given by the formula.

For $k=54$, six solutions may be derived from those for $k=2$, while three others are primitive.

14. The main query remains: (1) Are there further solutions for $k=3$? In view of the remarks of §11, and comparison with, for example, the cases $k=12$, 60, 66, 78, 93, this lack of solutions is less surprising than at first seems apparent, and the hope remains that a more refined and extended search may be successful.

Other queries that may be put are: (2) Does a solution exist for $k=2$, not given by the formula (15)? (3) Do solutions exist for the cases $k=30$, 33, 39, 42, 75, 84, 87, all of form $9\lambda \pm 3$?

In connection with these values of k, it is of interest to search for solutions with $k'=8k$, *i.e.* 240, 264, 312, etc.

Concerning the number of solutions found with $l \leqslant 3164$ we may ask: (4) Can an explanation be given for the irregularity of distribution of solutions with k? In particular: (5) Why have $k=83$ and $k=90$, for example, so many as 13 and 14 solutions? The preponderance of solutions for k a cube is notable: $k=1$, 8, 27, 64 have respectively 22, 18, 19, 19 primitive solutions.

[*Note added in proof.* An extension of the search to $a=3200$ has yielded two further solutions:

$$-3167^3+3166^3+311^3=64$$

$$3200^3-3168^3-991^3=97.]$$

Table

Solutions of $x^3+y^3+z^3=k$.

k	x	y	z	t†
1	1	0	0	0
	9	−8	−6	+1
	−12	10	9	−1
	−103	94	64	
	144	−138	−71	+2
	−150	144	73	−2
	172	−138	−135	
	−249	235	135	
	−495	438	334	
	505	−426	−372	
	577	−486	−426	
	729	−720	−242	+3
	−738	729	244	−3
	904	−823	−566	
	1010	−812	−791	
	1210	−1207	−236	
	−1544	1537	368	
	−1852	1738	1033	
	−1988	1897	1010	
	2304	−2292	−575	+4
	−2316	2304	577	−4
	3097	−2820	−1938	
2	1	1	0	0
	7	−5	−6	1
	49	−47	−24	2
	163	−161	−54	3
	385	−383	−96	4
	751	−749	−150	5
	1297	−1295	−216	6
	2059	−2057	−294	7
	3073	−3071	−384	8
3	1	1	1	
	−5	4	4	
6	2	−1	−1	
	65	−58	−43	
	236	−235	−55	
	644	−637	−205	
7	2	−1	0	
	−105	104	32	
	−169	168	44	

k	x	y	z
8	−16	15	9
	−34	33	15
	41	−40	−17
	−89	86	41
	−127	106	95
	−150	141	83
	−385	345	252
	−466	459	165
	−873	825	470
	995	−947	−514
	−1312	1293	459
	−1985	1808	1241
	−1987	1740	1371
	−2448	2189	1611
	2840	−2831	−601
	2883	−2526	−1987
	−2908	2671	1769
	−2920	2377	2255
	Also 17 derived solutions.		
9	2	1	0
	217	−216	−52
10	2	1	1
	4	−3	−3
	−171	141	130
	683	−650	−353
11	3	−2	−2
	258	−212	−197
	−641	619	297
	843	−695	−641
12	−11	10	7
15	2	2	−1
	−46	44	23
	332	−265	−262
16	1626	−1609	−511
	Also 7 derived solutions.		
17	2	2	1
	−52	50	25
	135	−111	−103
	492	−391	−390
	−558	473	408
18	3	−2	−1
	−123	101	94
	−218	215	75
	1671	−1373	−1276

k	x	y	z
19	3	−2	0
	19	−16	−14
	−77	76	26
	−95	91	47
20	3	−2	1
	−56	55	21
	156	−137	−107
	−275	256	159
	391	−387	−122
	1986	−1937	−827
	2833	−2816	−741
21	16	−14	−11
	−86	85	28
	−101	97	49
	445	−401	−287
24	Two derived solutions.		
25	3	−1	−1
	1167	−1159	−319
	−2683	2357	1839
26	3	−1	0
	−312	297	161
	−469	468	87
	−2107	2106	237
27	6	−5	−4
	19	−18	−10
	46	−37	−36
	−60	59	22
	115	−114	−34
	159	−131	−121
	−186	184	59
	340	−309	−214
	−358	354	115
	378	−340	−245
	−414	334	323
	−771	733	401
	−1130	915	878
	1188	−1156	−509
	−1403	1259	915
	1568	−1533	−632
	−1661	1652	420
	2206	−1954	−1485
	2434	−2325	−1228
	Also 15 derived solutions.		

† These are values of t in (14) or (15).

Solutions of $x^3+y^3+z^3=k$—*continued.*

k	x	y	z
28	3	1	0
	−17	14	13
	−59	56	31
	1541	−1526	−473
	2269	−2268	−249
29	3	1	1
	4	−3	−2
	−20	18	13
	235	−233	−69
30	No solution found.		
33	No solution found.		
34	3	2	−1
	5	−4	−3
	−6	5	5
	110	−109	−33
	−120	119	35
	147	−121	−112
	−508	447	347
	1557	−1555	−244
35	3	2	0
	14	−13	−8
	1154	−1120	−509
36	3	2	1
	4	−3	−1
	−75	71	40
	272	−269	−87
37	4	−3	0
	−56	50	37
	445	−444	−84
38	4	−3	1
	−27	25	16
39	No solution found.		
42	No solution found.		
43	3	2	2
	9	−7	−7
	−13	12	8
	−52	51	20
	837	−823	−307
44	8	−7	−5
45	4	−3	2
	−552	533	256
	−2369	2025	1709
46	3	3	−2
	−29	26	19
	815	−758	−473
47	−8	7	6
	31	−30	−14
	63	−50	−50
	−141	139	49
48	31	−26	−23
	Also 4 derived solutions.		
51	−796	659	602
52	No solution found.		
53	3	3	−1
	5	−4	−2
	−240	237	80
	2315	−2263	−935
	−2370	2141	1518
54	12	−11	−7
	−371	353	192
	1998	−1967	−715
	Also 6 derived solutions.		
55	3	3	1
	4	−2	−1
	10	−9	−6
	29	−23	−23
	−110	103	62
	199	−191	−97
	−249	246	82
	−368	367	74
	−566	559	188
	2197	−2195	−307
	2406	−2337	−1052
56	22	−21	−11
	−47	42	31
	−672	559	505
	Also 3 derived solutions.		
57	4	−2	1
	−38	34	25
	193	−185	−95
	−383	382	76
	−575	568	190
	835	−833	−161
	−998	982	361
60	5	−4	−1
	1202	−1201	−163
61	5	−4	0
	−966	845	668
62	3	3	2
	4	−1	−1
	5	−4	1
	−34	27	27
	−43	41	22
	−441	434	159
	2903	−2744	−1561
63	4	−1	0
	7	−6	−4
	63	−58	−38
	67	−63	−37
	−161	146	102
64	6	−5	−3
	25	−22	−17
	−110	101	67
	−152	151	41
	249	−248	−57
	−393	337	282
	519	−482	−303
	−620	617	151
	−786	669	571
	−879	814	519
	−1017	1012	249
	−1121	1110	345
	1219	−972	−963
	−1223	1198	479
	1447	−1438	−383
	−1539	1224	1219
	1866	−1607	−1329
	1899	−1587	−1418
	3011	−2555	−2198
	Also 24 derived solutions.		
65	4	1	0
	−111	91	85
	−759	730	364
	793	−770	−348
	−929	903	403
66	4	1	1
69	5	−4	2
	26	−22	−19
	−379	377	95
	−403	398	134

Solutions of $x^3+y^3+z^3=k$—*continued.*

k	x	y	z
70	-21	20	11
	-64	63	23
	581	-487	-432
	-694	693	113
	-2359	2325	824
71	4	2	-1
	5	-3	-3
	23	-20	-16
	-24	23	12
	36	-33	-22
	351	-342	-148
	-391	389	97
	-412	407	136
	459	-391	-333
	-533	443	401
72	-10	9	7
	28	-27	-13
Also two derived solutions.			
73	4	2	1
	25	-24	-12
	-47	43	29
	344	-335	-146
74	No solution found.		
75	No solution found.		
78	-55	53	26
	2123	-2080	-829
79	35	-33	-19
	74	-66	-49
	711	-706	-196
80	Four derived solutions.		
81	-18	17	10
	418	-351	-310
	2638	-2368	-1719
	Also 2 derived solutions.		
82	14	-11	-11
	-1317	1188	847
83	4	3	-2
	6	-5	-2
	25	-23	-15
	-29	24	22
	43	-36	-32
	115	-96	-86
	-183	151	139
	382	-317	-288
	388	-333	-278
	510	-509	-92
	2227	-2220	-470
	-2648	2595	1030
	2932	-2844	-1301
84	No solution found.		
87	No solution found.		
88	5	-4	3
	17	-16	-9
	167	-135	-130
	Also 4 derived solutions.		
89	-7	6	6
	-1330	1321	362
	-2514	2036	1953
90	4	3	-1
	5	-3	-2
	6	-5	-1
	11	-9	-8
	-27	26	13
	-48	47	19
	56	-51	-35
	75	-73	-32
	-100	99	31
	-456	443	199
	465	-454	-191
	-604	603	103
	1165	-1056	-739
	-1803	1798	365
91	4	3	0
	6	-5	0
	-381	364	192
	910	-869	-460
	-1341	1332	364
92	4	3	1
	6	-5	1
	9	-8	-5
	123	-119	-56
	169	-156	-101
	225	-218	-101
	576	-563	-233
	750	-749	-119
93	7	-5	-5
	253	-248	-98
96	One derived solution.		
97	5	-3	-1
	-22	18	17
	216	-199	-130
	-295	291	101
	-730	705	338
	2313	-1966	-1684
98	5	-3	0
	-15	14	9
	-141	140	39
	2391	-2101	-1638
99	4	3	2
	5	-3	1
	6	-5	2
	-37	36	16
	-176	155	120
	-237	214	152
	458	-453	-146
	-766	756	259
	984	-893	-622
100	7	-6	-3
	190	-161	-139
	1870	-1797	-903

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