The Mathematics of Music - a closer look into frequencies and tuning systems

Introduction

I have always been fascinated by how mathematics relates to music. I enjoy both subjects; it was great fun learning about their connections! In this article, I will explore the world of tuning systems, from Pythagoras' frustration of finding a proper, acceptable sound 2,500 years ago to the modern, innovative and unusual alternatives in use today.

But first, a brief introduction to the fundamental connections between music and mathematics. For thousands of years, people worldwide have known about the seemingly unique, magical sound that is now called music. In Egypt's Middle Kingdom, temple musicians used simple melodies to please the Pharaohs. Unfortunately, these melodies have now become extinct. Meanwhile, in Ancient Greece, music was often used to educate and teach young children, using the lyre, *kithara* (early harp) and *aulos* (early oboe).



However, some may ask how this relates to mathematics. It has long been recognised that tuning the 12 notes of the chromatic scale so that all intervals sounded consonant was impossible: some intervals would always be out of tune. The main problem lay in finding the optimal tuning (i.e. one that had the minimum dissonance).

Pythagorean tuning

One Greek pioneer of mathematics, commonly known for his right-angled-triangle theorem, is credited with the first tuning system. Pythagoras (*c.* 570 - 495 BCE) decided to base his tuning system on the **3:2** ratio, starting on an arbitrary note, say A, and tuning up or down by fifths (3:2) until one arrived at the original note.

A late-15th-century drawing by Franchinus Gaffurius illustrates the legend of Pythagoras experimenting with different-sized hammers in a blacksmith. The legend tells us that Pythagoras noticed a dissonant ringing in a blacksmith's shop and went to explore the source. After trying some of the hammers, he realised that certain multiples of a hammer's length (e.g. 1.5) made them ring harmoniously.
https://upload.wikimedia.org/wikipedia/commons/thumb/a/a0/Gaffurio_Pythagoras.png/2000px-Gaffurio_Pythagoras.png

Here is a table of how this would work.

Note that A is generally defined as 440 Hertz or vibrations per second), and the "equation shows how the frequency was arrived at.

Note that cents (¢) are defined as how many " $\frac{1}{1200}$ *s" of an octave there are above A.*

Note	**Frequency (Hertz)**	**Cents (¢) above A**	**Equation**
A	440	0	440
E	660	$1200\log_2\left(\frac{3}{2}\right) \pmod{1200} \approx 701.955$	$440 \times \frac{3}{2}$
B (C♭)	990	$1200\log_2\left(\left(\frac{3}{2}\right)^2\right) \pmod{1200} \approx 203.91$	$440 \times \left(\frac{3}{2}\right)^2$
F♯ (G♭)	1485	$1200\log_2\left(\left(\frac{3}{2}\right)^3\right) \pmod{1200} \approx 905.865$	$440 \times \left(\frac{3}{2}\right)^3$
C♯ (D♭)	2227.5	$1200\log_2\left(\left(\frac{3}{2}\right)^4\right) \pmod{1200} \approx 407.82$	$440 \times \left(\frac{3}{2}\right)^4$
G♯ (A♭)	3341.25	$1200\log_2\left(\left(\frac{3}{2}\right)^5\right) \pmod{1200} \approx 1109.775$	$440 \times \left(\frac{3}{2}\right)^5$
D♯ (E♭)	5011.875	$1200\log_2\left(\left(\frac{3}{2}\right)^6\right) \pmod{1200} \approx 611.73$	$440 \times \left(\frac{3}{2}\right)^6$
A♯ (B♭)	7517.8125	$1200\log_2\left(\left(\frac{3}{2}\right)^7\right) \pmod{1200} \approx 113.685$	$440 \times \left(\frac{3}{2}\right)^7$
E♯ (F)	11276.71875	$1200\log_2\left(\left(\frac{3}{2}\right)^8\right) \pmod{1200} \approx 815.64$	$440 \times \left(\frac{3}{2}\right)^8$
B♯ (C)	16915.078125	$1200\log_2\left(\left(\frac{3}{2}\right)^9\right) \pmod{1200} \approx 317.595$	$440 \times \left(\frac{3}{2}\right)^9$
G	25372.6171875	$1200\log_2\left(\left(\frac{3}{2}\right)^{10}\right) \pmod{1200} \approx 1019.55$	$440 \times \left(\frac{3}{2}\right)^{10}$

Note	Frequency (hertz)	Cents (¢) above A	Equation
A	440	0	440
D	$293\frac{1}{3} = 293.\dot{3}$	$1200\log_2\left(\frac{2}{3}\right) \pmod{1200} \approx 498.045$	$440 \times \frac{2}{3}$
G	$195\frac{5}{9} = 195.\dot{5}$	$1200\log_2\left(\left(\frac{2}{3}\right)^2\right) \pmod{1200} \approx 996.09$	$440 \times \left(\frac{2}{3}\right)^2$
C (B♯)	$130\frac{10}{27} = 130.\dot{3}7\dot{0}$	$1200\log_2\left(\left(\frac{2}{3}\right)^3\right) \pmod{1200} \approx 294.135$	$440 \times \left(\frac{2}{3}\right)^3$
F (E♯)	$86\frac{74}{81} = 86.\dot{9}1358024\dot{6}$	$1200\log_2\left(\left(\frac{2}{3}\right)^4\right) \pmod{1200} \approx 792.18$	$440 \times \left(\frac{2}{3}\right)^4$
B♭ (A♯)	$57\frac{229}{243}$	$1200\log_2\left(\left(\frac{2}{3}\right)^5\right) \pmod{1200} \approx 90.225$	$440 \times \left(\frac{2}{3}\right)^5$
E♭ (D♯)	$38\frac{458}{729}$	$1200\log_2\left(\left(\frac{2}{3}\right)^6\right) \pmod{1200} \approx 588.27$	$440 \times \left(\frac{2}{3}\right)^6$
A♭ (G♯)	$25\frac{1645}{2187}$	$1200\log_2\left(\left(\frac{2}{3}\right)^7\right) \pmod{1200} \approx 1086.315$	$440 \times \left(\frac{2}{3}\right)^7$
D♭ (C♯)	$17\frac{1103}{6561}$	$1200\log_2\left(\left(\frac{2}{3}\right)^8\right) \pmod{1200} \approx 384.36$	$440 \times \left(\frac{2}{3}\right)^8$
G♭ (F♯)	$11\frac{8767}{19683}$	$1200\log_2\left(\left(\frac{2}{3}\right)^9\right) \pmod{1200} \approx 882.405$	$440 \times \left(\frac{2}{3}\right)^9$
C♭ (B)	$7\frac{37217}{59049}$	$1200\log_2\left(\left(\frac{2}{3}\right)^{10}\right) \pmod{1200} \approx 180.45$	$440 \times \left(\frac{2}{3}\right)^{10}$

In **equal temperament**, our modern system, E♭ and D♯ are equal. In Pythagorean tuning, E♭ and D♯ are where the ascending fifths and descending fifths meet. However, the nature of this system means that E♭ is 588.27 cents above A, and D♯ is 611.73 cents above A, so they are not equal. The difference between these two intervals is **23.46 cents**. This is called the Pythagorean comma and is arbitrarily the difference between a given interval from its perfect counterpart, such as 3:2, where the difference can be perceived as *dissonant* to the ear.

Melodically, Pythagorean tuning is satisfactory. However, when harmony is included, the whole system falls apart due to noticeable variations in the size of the semitones.

As a result, Pythagoras' system and theory are no longer in use, due to striking dissonances known as *wolf tones*. Nevertheless, they provided a stepping stone for other musicians and mathematicians to create and put forward new formulae, trying to find the most accurate and convenient tuning system. During the 1,900 years between Pythagoras' tuning and the rise of equal temperament, other systems were put forward, each with its disadvantages and problems.

Meantone temperament

This is a generalised term to refer to a set of tuning systems such as quarter-comma meantone and third-comma meantone. Meantone temperament had its roots in the early 16th century. It was first alluded to in 1496 by Franchino Gafori in his treatise called "*Practica Musica*". In "*De musica libri septem*", a 1577 book by Francisco de Salinas, he refers to what is now known as quarter-comma meantone, two-seventh-comma meantone and third-comma meantone.

Quarter-comma meantone involved tuning all major thirds (i.e. C to E, D♭ to F, D to F♯ etc.) to the ratio **5:4**, thereby flattening all the fifths slightly.

Each of the fifths has the ratio

$$\sqrt[4]{5} : 1 \approx 1.49535 : 1$$

compared with

$$1.5 : 1$$

for perfectly-tuned fifths.

This system is called **quarter-comma meantone** since it flattens the fifth by **a quarter of a comma** or around **6 cents**. All major sixths are also sharp by a quarter of a comma.

This system worked, so long as *enharmonic* or **equal pitches** such as C♯ and D♭ were not used. If both C♯ and D♭ needed to be used in the same composition, the performer would need to use a keyboard with a double-black key. The reason for this was similar to Pythagorean tuning; when the "ascending thirds" sequence met the "descending thirds" sequence, the ratios involved would not coincide exactly. As a result, the interval F♯-D♭ is nearly **two commas** out of tune and is unacceptable, whether in the Middle Ages or now.

Equal temperament

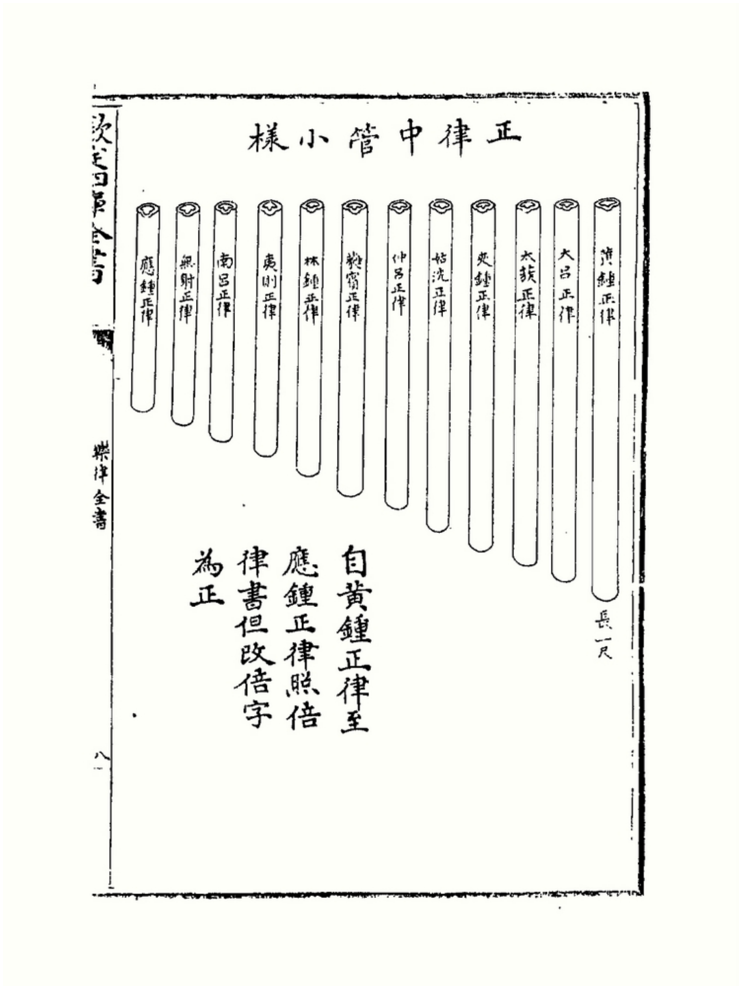
The modern tuning system we know today is called **12-tone equal temperament** or 12-TET. Its discovery is often credited to Zhu Zaiyu (1584) in Asia and Simon Stevin (1585) in Europe. This involved splitting the octave into 12 semitones, based on exponential frequencies, as shown in the table below.

Note	**Frequency (Hz)**	**Cents (¢) above A**	**Equation**
A	440	0	440
B♭ (A♯)	466.16376151808991640...	100	$440 \times \sqrt[12]{2}^{1}$
B	493.88330125612411183...	200	$440 \times \sqrt[12]{2}^{2}$
C	523.25113060119726935...	300	$440 \times \sqrt[12]{2}^{3}$
C♯ (D♭)	554.36526195374419249...	400	$440 \times \sqrt[12]{2}^{4}$
D	587.32953583481512052...	500	$440 \times \sqrt[12]{2}^{5}$
E♭ (D♯)	622.25396744416182147...	600	$440 \times \sqrt[12]{2}^{6}$
E	659.25511382573985947...	700	$440 \times \sqrt[12]{2}^{7}$
F	698.45646286600776889...	800	$440 \times \sqrt[12]{2}^{8}$
F♯ (G♭)	739.98884542326879786...	900	$440 \times \sqrt[12]{2}^{9}$
G	783.99087196349858817...	1000	$440 \times \sqrt[12]{2}^{10}$
A♭ (G♯)	830.60939515989027704...	1100	$440 \times \sqrt[12]{2}^{11}$

At first glance, it seems like A and E♭ would sound harmonious together since they are 600 cents apart which is half of 1200. However, a closer inspection of the process involved in creating the frequencies reveals that the ratio of the frequencies of A and E♭ is $\sqrt{2}$ which is an irrational number. As a result, this means that, as Pythagoras found out, A and E♭ will clash and sound disharmonious. This interval is called a "tritone" and was even **forbidden** in the medieval era.

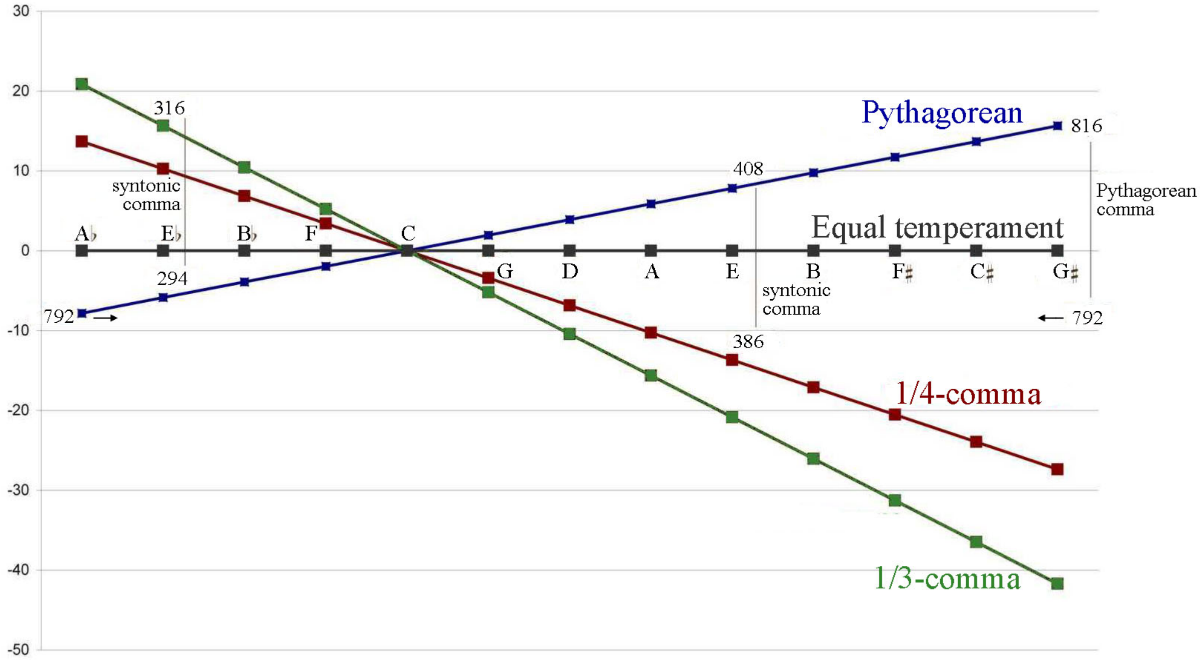
Contrary to this, one pleasant-sounding interval is the interval from A to E or a perfect fifth. Even though they are 700 cents apart, the ratio of the frequencies of A and E is approximately 1.4983, just under 1.5. This means that the vibrations and wave patterns would coincide and sound delightful, like the hammers ringing together.

The advantage of equal temperament is that all intervals would sound more or less in tune; there would not be any harsh, dissonant intervals. This was the first system to solve a long-standing problem of suitable harmony in fretted instruments and keyboard instruments.

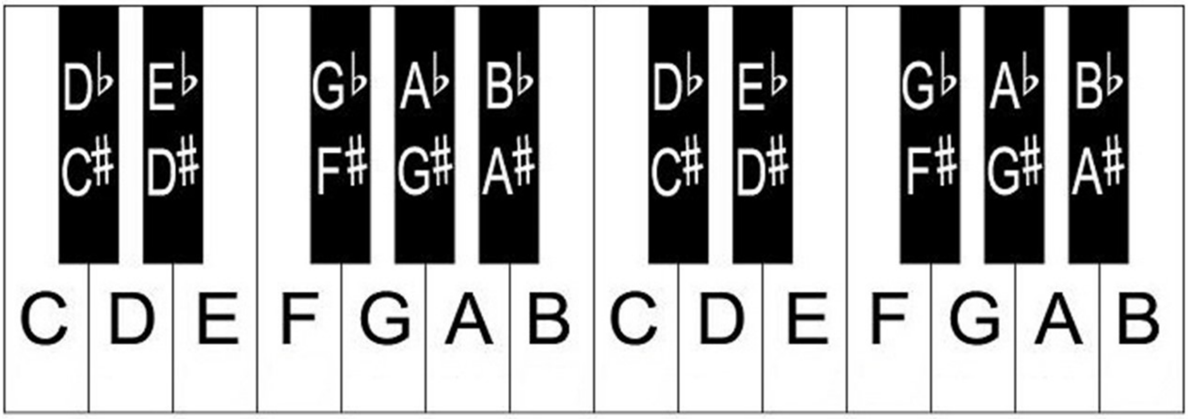


Zhu Zaiyu's diagram of his early tuning system, *using pan-pipes. Most scholars regard this to be describing equal temperament.*
https://upload.wikimedia.org/wikipedia/commons/f/f2/%E4%B9%90%E5%BE%8B%E5%85%A8%E4%B9%A6%E5%85%A8-1154.jpg

Comparison of tuning systems



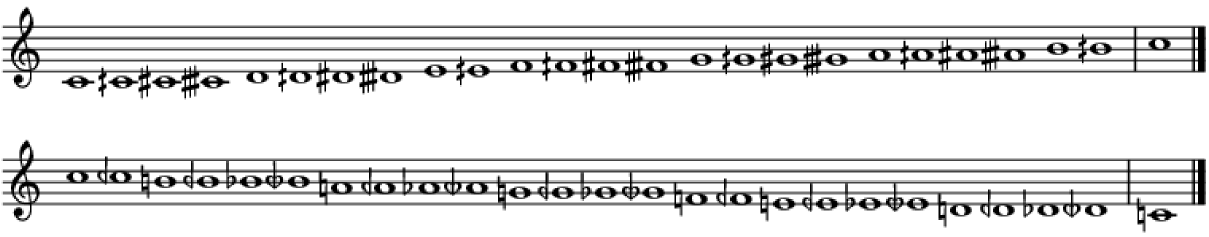
This diagram shows the relationship between the four main tuning systems: quarter-comma meantone, third-comma meantone, equal temperament and Pythagorean tuning. As can be seen in the diagram, there are considerable discrepancies between the tuning systems, which explains the noticeable differences when switching between them.



As shown in the diagram above, every single combination of notes needed to sound in tune. However, early tuning systems always had a set of intervals that would be very dissonant (such as mean-tone temperament and Pythagorean tuning). For example, when the chord D-F-A was played on an instrument tuned to Pythagorean tuning, the notes would clash. The resulting sound would be undesirable.

Microtonal tuning systems

As the 20th century progressed, composers experimented with unusual-sounding systems, such as **24-tone equal temperament** (24-TET). In 1823, Heinrich Richter discovered 24-TET; many other composers have experimented with it since. 24-TET includes all the notes of the usual 12-tone scales but adds notes in between (called quarter-tones).



This is the 24-tone scale. It includes the 12 chromatic notes, plus 12 extra notes in between. Special symbols were added to signify half-sharps, three-half-sharps, half-flats and three-half-flats.

Traditional Persian scales and instruments make use of the quarter-tone scale. Modern composers have also used it, such as Charles Ives' 3 quarter-tone pieces for two pianos.

In addition to 24-TET, composers have looked at 19-TET, 31-TET and 53-TET. These systems work in virtually the same way.

19-TET splits the octave up into 19 intervals in the ratio $\sqrt[19]{2}$,

whilst 31-TET splits the octave up into 31 intervals in the ratio $\sqrt[31]{2}$

and 53-TET splits the octave up into 53 intervals in the ratio $\sqrt[53]{2}$.

The advantage of these scales is that they allow for greater precision in the tuning of perfect fifths than normal **12-TET**.

Also, a relatively simple formula can be used to calculate the correct frequencies:

$$440 \times \sqrt[\frac{T}{n}]{2}$$

$$\text{where } T = \text{ the number of divisions of the octave}$$

$$\text{and } n = \text{ the } n\text{th overtone in the sequence}$$

Conclusion

Next time, when you are tuning your musical instrument, you will now realise how important mathematics has been in developing the history of tuning over the years!

I hope that this report has helped you appreciate the fascinating and complex relationships between mathematics and music. Beneath the surface of instruments and composition, there lies a whole mathematical discovery and discussion to be found.

I also hope that this essay will encourage you to explore more about tuning, harmonic intervals and their relationship with mathematics.

This image shows a tuning fork pitched at 659 Hertz (or approximately the note E at equal temperament).

https://upload.wikimedia.org/wikipedia/commons/b/b2/TuningFork659Hz.jpg

References and further reading

- https://en.wikipedia.org/wiki/Musical_tuning#Tuning_systems - a list of traditional and historical tuning systems.

- https://en.wikipedia.org/wiki/Music_and_mathematics#Frequency_and_harmony - a closer look at microtonal intervals and tuning systems.

- https://en.wikipedia.org/wiki/Microtonal_music

- The *Encyclopædia Britannica* has an insightful look into the complex mathematical concepts of harmony, tuning and dissonances in its *Macropædia* article entitled *The Art of Music* (Volume 25, pages 509 to 511)

- https://youtu.be/izFgt2tZ0Oc?t=21 - Charles Ives' Quarter Tone Piece no.1 for two pianos