Imaginary Numbers: A Journey From the Ridiculed to the Indispensable

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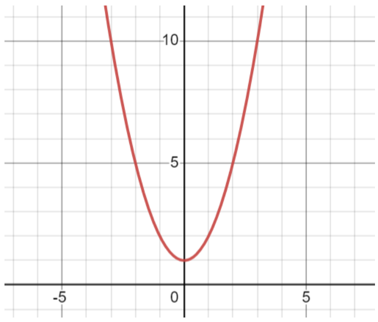
For many years, imaginary numbers were treated with derision – mathematicians that found them in their work despaired at the broken mathematics before them. When the name "imaginary" was coined by Descartes, it was in attempt to delegitimise them. You may ask, if many of mathematics' greatest minds did not accept imaginary numbers, why should you? It is worth considering that the negative numbers that are now taught to primary school students, were once treated with the same condescension. It seems this cycle of scepticism is an inherent part of inventing new mathematics.

Take a simple quadratic, similar to those that GCSE maths students may get: $x^2 + 1 = 0$. To solve the equation for x, it can be rearranged into the form $x^2 = -1$. Here, the reader is presented with an issue. The solution for x would require taking the square root of negative one. But what is that? It can't be 1, because $1 \times 1 = 1$, but $(-1) \times (-1) = 1$ too. The graph showing $y = x^2 + 1$ would indicate that there are no solutions, as it never crosses the x-axis (where y would be equal to zero). Indeed, that is what mathematicians of the 1500s believed. However, today's mathematicians will explain that solutions to this equation . The secret lies in a type of number that cannot fit on the Cartesian coordinate system, called the imaginary numbers.



The graph of y=x²+1

Imaginary numbers are numbers that when multiplied together, give a negative result. They are written in terms of i, which is called the imaginary unit, and is equivalent to √(-1). This means that in the same way that $2 \times 2 = 4$, $i \times i = -1$. If this seems slightly odd, or even downright insane, don't worry. Many esteemed mathematicians were unconvinced of these numbers as well. So, if so many people could not accept imaginary numbers, how did they come to be as widely adopted and utilised as they are today?

Despite imaginary numbers being the square root of negative numbers, they were first used in the creation of the cubic formula: a method of finding the solutions to equations in the form $ax^3 + bx^2 + cx + d = 0$. The quadratic formula ($x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$), which is used to find the solutions to equations of the form $ax^2 + bx + c = 0$, was well established and had been used since the Babylonians, but by the 16th century, mathematicians were no closer to finding a cubic formula, and many considered it an impossible task.

Enter Scipione del Ferro, an Italian mathematician, who in the early 1500s discovered the formula for a special version of the cubic equation, of the form $x^3 + cx = d$ where c, d > 0. In modern day algebraic notation, his formula would be written as

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}.$$

This was a significant breakthrough in finding a general solution to the cubic equation, and a huge accomplishment, yet del Ferro kept his formula secret, as was common in mathematical circles at this time. Mathematicians gained fame and prestige not by publishing their work, but by challenging other mathematicians to so called "mathematics duels". Each would send the other a "challenge gauntlet": a list of challenges for them to solve, and the person who could solve the most problems won the duel. Those who won frequently garnered more students and retained the support of wealthy benefactors who sponsored their research. As such, del Ferro kept his discovery a secret until his final days, when he shared the technique with his student, Antonio Fior.

Armed with del Ferro's formula, Fior was sure he was invincible, and challenged a significantly stronger mathematician, Niccolò Fontana (also known as Tartaglia) to a mathematical duel. Upon hearing that Fior had a formula to solve the depressed cubic (cubic equations that do not contain the x^2 term), Tartaglia set out to find one himself. Being successful, he completed all of the questions given to him by Fior in only two hours, while Fior could answer none himself.

The news of Tartaglia's victory spread, together with his ability to solve the cubic. This alerted another mathematician, Gerolamo Cardano, and so began one of the greatest mathematical rivalries. Believing that finding a cubic formula was impossible, Cardano was shocked by the news of Tartaglia's accomplishment, and, finding himself unable to replicate the formula on his own, he began a pressure campaign to convince Tartaglia to share his technique. Tartaglia ultimately relented, sharing his method with Cardano along with an oath of secrecy.

Regardless, in 1545 Cardano went on to published *Ars Magna*, which contained an extended version of Tartaglia's method to find the solutions for 13 different types of cubic equations, written so that the coefficients were always positive (as many were sceptical of negative numbers). This was accomplished by converting them into depressed cubics, which could be solved using Tartaglia's technique. However, with some equations, this formula returned an odd phenomenon: the square root of negative numbers. Cardano immediately dismissed this as impossible, and simply did not include these in his summary.

Mathematics had to wait for Rafael Bombelli, who was the first to accept the square root of negative numbers, and to treat them as their own type of number. He found that if you continued the calculations with these strange new numbers, and didn't simply stop when they were introduced, Cardano's method returned the real solution to the cubic equation, with the negative roots (imaginary parts) cancelling out.

Despite the successful applications of these imaginary numbers, Bombelli viewed them with suspicion, and ultimately disregarded them as merely a useful hack. They continued to be treated with scepticism, and it was not until the work of Euler (1707-1783) and Gauss (1777-1855) that they were widely accepted. Euler gave them the symbol i, for imaginary, and therefore all the other numbers are given the name real numbers. The combination of the two of them (e.g. $2 + 3i$) are complex numbers.

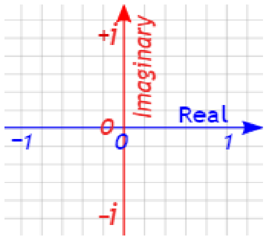
This leads to the question, where are the complex numbers? They do not fall anywhere on our number line. How can we interact with them? The answer here, lies in the complex plane.

The imaginary unit, i, has an interesting property that appears when you begin to raise it to positive integer exponents.

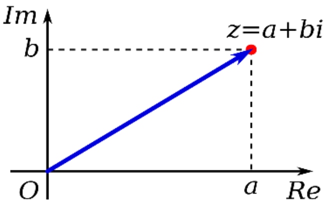
$$i^1 = i \qquad\qquad i^5 = i$$
$$i^2 = -1 \qquad\qquad i^6 = -1$$
$$i^3 = -i \qquad\qquad i^7 = -i$$
$$i^4 = 1 \qquad\qquad i^8 = 1$$

Hopefully, you can see that a cyclic pattern emerges when raising i to increasingly higher powers. The powers of 1 through 4 show the first cycle of the pattern $(i, -1, -i,\ 1)$. This is similar to the pattern that emerges when raising -1 to increasing powers, which creates the pattern 1, -1, 1, -1, …

This provides us with the key insight as to how complex numbers are represented. If you imagine a number on the number line being represented by an arrow from zero to that number, then multiplying by -1, or i^2, is a 180° rotation of that arrow. This results in the arrow facing the opposite direction, and pointing to a negative result with the same magnitude. Multiplying by -1 again, would return the arrow back to its original position, as it represents multiplying by $(-1)^2$ which is equal to 1. Using the same logic for i, some combination of two equal rotations will give the same effect as multiplying by i^2, which causes a rotation of 180°. The transformation that would do this is a rotation by 90°.



The complex plane



The representation of the complex number a+bi on the

As a result of this, the imaginary numbers can be thought as having their own number line, which runs perpendicular to that of the real numbers. Every multiplication by i causes a rotation of the arrow by 90°, so that after four multiplications it is back in the original position. This extension of the number line means it is no longer a single line, and is referred to as the complex plane. Complex numbers are represented as a coordinate that is the combination of their real part and their imaginary part, as on the Cartesian plane.

The beauty of complex numbers stems from the idea of closure. A set of numbers is closed under a particular operation, if the result of that operation is always an element of the set. For example, the set of natural numbers, which is all integers greater or equal to one, is closed under addition, but not under subtraction. Adding two integers together results in a larger integer, so it must be in the set, therefore it is closed. However, if a larger integer is subtracted from a smaller integer (e.g. $4-6$), the result is not a natural numbers, so the set is not closed under subtraction. The real numbers are closed under addition, subtraction, multiplication and division, but not when raised to all powers (as fractional powers are an alternative way of writing roots). The square root of a negative numbers is not a real number, it is imaginary. However, the complex numbers possess the property of closure for addition, subtraction, multiplication, division, and raising them to powers. Complex numbers are closed under every operation that can be applied to them – even the square roots of i are complex numbers ($\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$).

While this may be satisfying, it does not answer the question of how complex numbers are utilised in the real world. Applications of complex numbers appear within a wide variety of physics topics, including the wave function in quantum mechanics, which provides the probability distribution of the possible positions of a particle at a specific time. In addition, imaginary numbers are a valuable tool that have revolutionised the world of signal processing. One of the most common uses is in modelling how electric circuits behave. The main concept behind this is Euler's formula

$$e^{ix} = \cos x + i \sin x$$

from which Euler's identity ($e^{ix}+1=0$), sometimes called the most beautiful formula in all of mathematics, stems. Complex numbers are a necessary part of calculations with AC (alternating current) circuits, because of phase. Phase represents how far a periodic function has shifted: it can be thought of as how offset the graph would be. Imaginary numbers become pertinent when trying to add trigonometric functions with the same frequency, but different phase. This often occurs in analysis of AC circuits, when voltages with different phases are put together in one circuit. Before further calculations can be performed on it, the voltage needs to be simplified. The basic principle, is that the real part of e^{ix} is equal to $\cos x$. This means that

$$\cos(x+a) + \cos(x+b) = Re\left(e^{i(x+a)}\right) + Re\left(e^{i(x+b)}\right) = Re(e^{ix}(e^{ia} + e^{ib}))$$

which is simpler to rearrange into one identity, and thus provides a method for adding trigonometric functions with different phases.

Just like negative numbers, the mathematics of imaginary numbers has journeyed from complete disregard, through mild scepticism, to where they are today, a necessary player within the realm of quantum mechanics and electric circuits. They represent the closure of the number system under all types of operations. While they may be called imaginary, they are anything but!