Extending the Factorial

Introduction

One of the most interesting aspects of mathematics is the extension of certain concepts into larger sets of numbers. For instance, we are all initially taught to think of exponentiation as repeated multiplication. For example: $2^3 = 2 \times 2 \times 2$. This definition of exponentiation as the repeated multiplication of the base only makes sense for whole number exponents. However, mathematicians then figured out ways of making sense of rational, irrational, imaginary and even matrix exponents. I believe that this way of thinking leads to the discovery of the most beautiful results in all of mathematics. One such beautiful result is Euler's identity: $e^{i\pi} = -1$.

Mathematicians thought of another concept that could produce interesting results when extended into real numbers from natural numbers: factorial of a natural number n, denoted as $n!$.

The factorial of n is defined in natural numbers as follows:

$$n! \coloneqq n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

$$so, n! = n \times (n-1)!$$

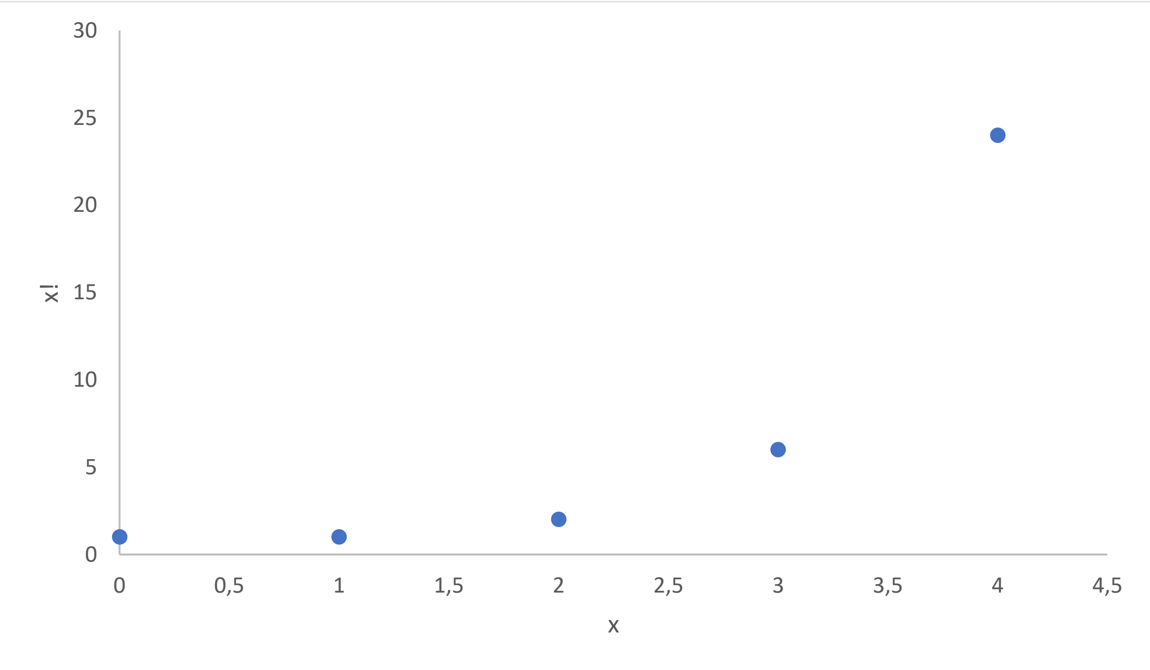
This recursive formula allows us to figure out the factorials of natural numbers. Plugging in $n = 1$, we get $1! = 1 \times (1-1)!$, which means that $0!$ is equal to 1. In general, if the factorial of a natural number is already known, the next or the previous natural number's factorial can easily be calculated using the recursive formula.

Factorials show up in the series expansions of various functions, combinatorics, probability, computer science and in many more fields of mathematics and science. One of the most important and popular applications of the factorial is in calculating the number of arrangements of multiple items. n many items can be arranged in $n!$ many ways. Another use of the factorial is in the estimation of the number e, using the formula: $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \approx 2.718281828459$.

After being told perpetually that the definition of $n!$ only holds for natural numbers in school, I was blown away when I first found out that it was possible to extend it to real numbers. Being my first recognition of mathematical beauty, this was an especially special discovery of mine, because it greatly increased my appreciation and passion for mathematics. I started to improve my understanding of calculus to fully comprehend this concept, growing a larger and larger passion for mathematics along the way.

Deriving the Generalised Definition of the Factorial

When the values of $n!$ are plotted on a graph, discrete points that are not connected by a curve are produced. One way of connecting the dots is making use of a function. The discrete points form the following visual:



Generalising the factorial corresponds to defining a function (which I will denote as $\Pi(x)$), such that it connects the dots on the graph with a curve. The factorial function should have the following properties:

- $\Pi(x) = x!$
- $\Pi(x) \equiv x\Pi(x-1)$
- The function must take all values of x in real numbers (except for possible values where the function is undefined).

In order to derive $\Pi(x)$, we will start by considering the following parameterised integral function:

$$I(s) = \int_0^{\infty} e^{-st}\, dt$$

The next step is to differentiate this function repeatedly and look for a pattern. We will make use of the Leibniz integral rule to differentiate under the integral sign (also known as Feynman's integration trick).

$$\Rightarrow \frac{dI(s)}{ds} = \int_0^{\infty} \frac{\partial}{\partial s}(e^{-st})\, dt = \int_0^{\infty} -te^{-st} dt$$

$$\Rightarrow \frac{d^2 I(s)}{ds^2} = \int_0^{\infty} \frac{\partial}{\partial s}(-te^{-st}) dt = \int_0^{\infty} t^2 e^{-st}\, dt$$

$$\Rightarrow \frac{d^3 I(s)}{ds^3} = \int_0^{\infty} \frac{\partial}{\partial s}(t^2 e^{-st}) dt = \int_0^{\infty} -t^3 e^{-st}\, dt$$

$$\vdots$$

$$\therefore \frac{d^n I(s)}{ds^n} = \int_0^{\infty} (-1)^n t^n e^{-st} dt = (-1)^n \mathcal{L}\{t^n\}$$

The expression for the n$^{\text{th}}$ derivative of $I(s)$ is equivalent to $(-1)^n \mathcal{L}\{t^n\}$, where $\mathcal{L}\{t^n\}$ is the Laplace transform of t^n. In order to complete our derivation, we must first find an expression for $\mathcal{L}\{t^n\}$. We will make use of $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, and the series expansion of the functions $\frac{1}{1-x}$ and e^{ax}.

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} = \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{a^n t^n}{n!}\right\}, and\ \frac{1}{s-a} = \frac{1}{s} \cdot \frac{1}{1-\frac{a}{s}} = \frac{1}{s}\sum_{n=0}^{\infty} \left(\frac{a}{s}\right)^n$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{a^n t^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}}$$

Assuming that the Laplace transform and the summation are interchangeable:

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \mathcal{L}\{t^n\} = \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}}$$

Since both sides are sums with the same indices, we can conclude that:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

In the beginning of the derivation, we found out that the function $I(s)$'s n[th] derivative is $(-1)^n\mathcal{L}\{t^n\}$, which means that:

$$\Rightarrow (-1)^n\mathcal{L}\{t^n\} = \frac{(-1)^n n!}{s^{n+1}} = (-1)^n \int_0^\infty t^n e^{-st} dt$$

$$\Rightarrow \frac{n!}{s^{n+1}} = \int_0^\infty t^n e^{-st} dt$$

Substitution: $u = st,\ du = sdt$

$$\Rightarrow \frac{n!}{s^{n+1}} = \frac{1}{s^{n+1}} \int_0^\infty u^n e^{-u} du$$

$$\therefore n! = \int_0^\infty t^n e^{-t} dt$$

We have finally derived an expression that represents the factorial as an integral. Therefore, we can now conclude that our generalised factorial function, $\Pi(x) = \int_0^\infty t^x e^{-t} dt$ can be used to calculate the factorial of any real number. $\Pi(x)$ also satisfies the conditions we have listed before (I will not go through the process of proving these properties, because the proofs do not add much to the final conclusion).

<u>Note</u>: This derivation can be made more rigorous by justifying the interchange of limits and providing the conditions under which the infinite sums converge.

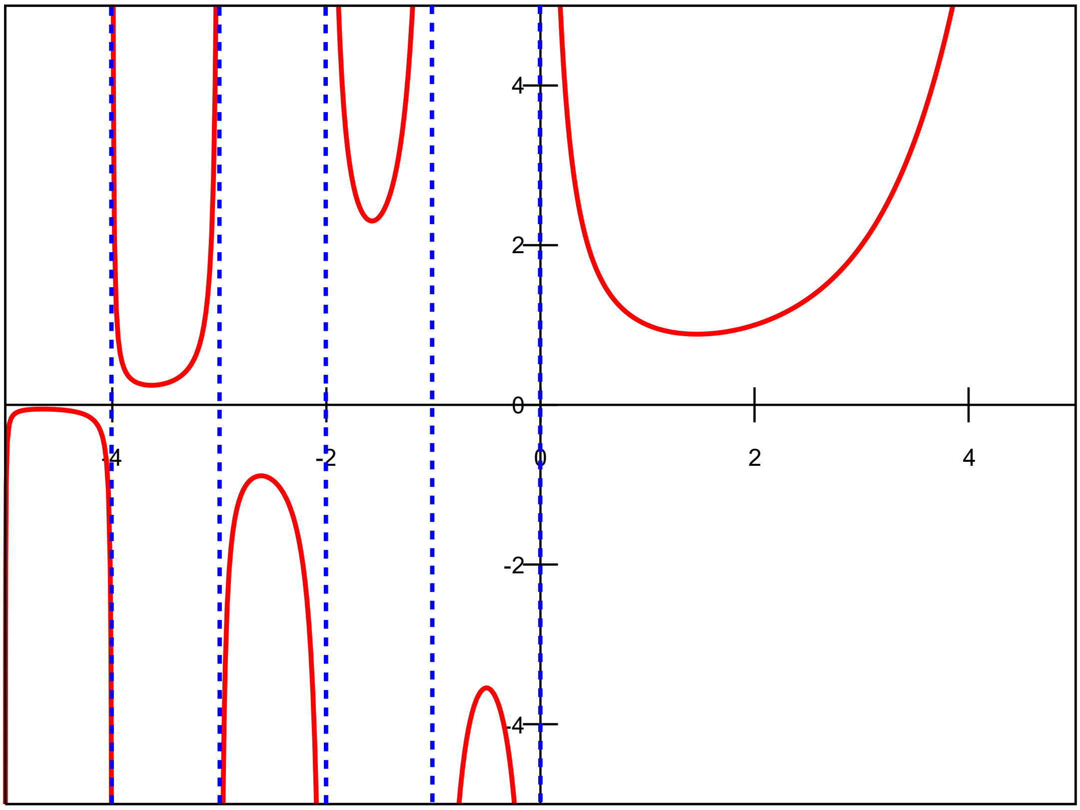
A slightly altered version of $\Pi(x)$ that is more commonly used in the mathematical community, known as the Gamma function, $\Gamma(x)$, is defined as follows:

$$\Gamma(x) = \int_0^\infty t^{n-1} e^{-t} dt$$

So $\Gamma(x) = \Pi(x-1)$, and $\Gamma(x) = (n-1)!$ for whole numbers.

It is still often debated which of the two functions is more sensible to use to simplify equations and identities, but the Gamma function is overwhelmingly more commonly used than the Pi function.

Now that we have derived the Gamma function, we can finally graph it to connect the dots on the previous plot with a curve. The Gamma function's graph looks like this:



I remember being surprised to see this graph when I asked Wolfram Alpha to graph $x!$. I had no idea why the graph looked like this, because I did not know anything about the gamma function back then. It was impossible for me to make any sense of what I saw as a first year IGCSE student.

The graph shows that the Gamma function is clearly showing asymptotic behaviour at negative integers. This is because the area under the graph of $t^{x-1}e^{-t}$ is infinite between the $t = 0$ and $t \to \infty$, for x being a negative integer. Therefore, the Gamma function is undefined for these values of x.

All the odd negative integer values of $\Gamma(x)$ diverge to infinity, while the even negative integer values of $\Gamma(x)$diverge to negative infinity. It is also apparent that there are no points at which $\Gamma(x) = 0$.

We can now use the Gamma function to determine the values of different factorials. This can be done using alternative representations of the Gamma function, and the formulae that we will derive using these representations.

Alternative Representations of the Gamma Function

Another peculiar thing about the Gamma function is that it can be represented in many different forms. Let us take a look at different representations of the Gamma function.

1. Gauss Representation:

Using the limit definition of the exponential function, the integral representation of the Gamma function can be transformed into the following:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \int_0^\infty t^{x-1} \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n dt$$

Assuming that the limit and integration are interchangeable, we get:

$$\Gamma(x) = \lim_{n\to\infty}\int_0^\infty t^{x-1}\left(1-\frac{t}{n}\right)^n dt$$

After doing n+1 iterations of integration by parts and evaluating the anti-derivative at the limits of integration (which I will not demonstrate to avoid unnecessarily prolonging this derivation), we get:

$$\Gamma(x) = \lim_{n\to\infty}\frac{n^x}{x}\cdot\frac{1\cdot 2\cdots(n-1)\cdot n}{(x+1)\cdot(x+2)\cdots(x+n-1)\cdot(x+n)}$$

Expressing the product in capital pi notation, the following final form is obtained:

$$\therefore \Gamma(x) = \frac{1}{x}\lim_{n\to\infty} n^x \prod_{k=1}^{n}\frac{k}{x+k}$$

2. Euler Representation:

The number n can be represented in the following way:

$$n = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1} = \prod_{k=1}^{n-1}\frac{k+1}{k}$$

Multiplying the product with the term at n and its reciprocal:

$$n = \prod_{k=1}^{n-1}\frac{k+1}{k}\cdot\frac{n+1}{n}\cdot\frac{n}{n+1} = \frac{n}{n+1}\prod_{k=1}^{n}\frac{k+1}{k}$$

Plugging this representation of n into the Gauss representation gives the Euler representation:

$$\Gamma(x) = \frac{1}{x} \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^x \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^x \prod_{k=1}^{n} \frac{k}{x+k}$$

$$\therefore \Gamma(x) = \frac{1}{x} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^x}{1 + \frac{x}{k}}$$

3. Weierstrass Representation

The final and most interesting representation of the Gamma function is due to Weierstrass. We will start with the Gauss representation, and write n^x as $\exp(x \ln n)$.

$$\Gamma(x) = \frac{1}{x} \lim_{n\to\infty} \exp(x \ln n) \prod_{k=1}^{n} \frac{k}{x+k}$$

$$\Gamma(x) = \frac{1}{x} \lim_{n\to\infty} \exp\,(xH_n - xH_n + x \ln n) \prod_{k=1}^{n} \frac{k}{x+k}$$

$$\textit{where } H_n \textit{ is the harmonic series, defined as } \sum_{k=1}^{n} \frac{1}{k}$$

Further algebraic manipulation yields:

$$\Rightarrow \Gamma(x) = \frac{1}{x} \lim_{n\to\infty} \exp\,(xH_n) \exp\,(x(\ln n - H_n)) \prod_{k=1}^{n} \frac{k}{x+k}$$

$$\Rightarrow \Gamma(x) = \frac{1}{x} \lim_{n\to\infty} \exp\,(x \sum_{k=1}^{n} \frac{1}{k}) \exp\,(x(\ln n - H_n)) \prod_{k=1}^{n} \frac{k}{x+k}$$

$$\Rightarrow \Gamma(x) = \frac{1}{x} \lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(\frac{x}{k}\right) \exp\,(x(\ln n - H_n)) \prod_{k=1}^{n} \frac{k}{x+k}$$

The limiting difference between the Harmonic series and the natural logarithm is equal to what is known as the Euler-Mascheroni constant, $\gamma = \lim_{n\to\infty} (H_n - \ln n)$.

In light of this information, we can conclude that:

$$\therefore \Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} e^{\frac{x}{k}} \left(1 + \frac{x}{k}\right)^{-1}$$

These different representations are going to be useful when deriving relationships involving the Gamma function, one of which is Euler's reflection formula.

Euler's Reflection Formula

Perhaps the most important and useful functional equation of the gamma function is Euler's reflection formula, which I will derive now using the Euler representation of the gamma function along with the infinite product representation of $\sin \pi x$.

We will begin by considering $\Gamma(x)\Gamma(1-x)$, which is equivalent to $(-x)\Gamma(-x)\Gamma(x)$.

$$\Rightarrow -x\Gamma(-x)\Gamma(x) = \frac{1}{x} \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^{x}}{1+\frac{x}{k}} \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^{-x}}{1-\frac{x}{k}}$$

$$\Rightarrow \Gamma(x)\Gamma(1-x) = \frac{1}{x} \prod_{k=1}^{\infty} \frac{1}{1-\frac{x^2}{k^2}}$$

This product is very similar to the infinite product representation of $\sin \pi x$, which is:

$$\sin \pi x = \pi x \prod_{k=1}^{\infty} 1 - \frac{x^2}{k^2}$$

Taking the reciprocal of both sides gives:

$$\frac{1}{\sin \pi x} = \frac{1}{\pi x} \prod_{k=1}^{\infty} \frac{1}{1-\frac{x^2}{k^2}}$$

Multiplying both sides by π gives:

$$\frac{\pi}{\sin \pi x} = \frac{1}{x}\prod_{k=1}^{\infty}\frac{1}{1-\frac{x^2}{k^2}}$$

In conclusion:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$\therefore \Gamma(x)\Gamma(1-x) = \pi\csc \pi x$$

We can use this result to calculate the value of $\left(-\frac{1}{2}\right)!$. Plugging in $x = \frac{1}{2}$ into Euler's reflection formula gives:

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \pi\csc\frac{\pi}{2} = \pi$$

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi, \qquad \therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

This surprising result enables us to determine other values of the gamma function using its recursive definition. For example:

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right), \ \therefore \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2},$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right), \qquad \therefore \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}, \ etc.$$

Gamma Function's Relation to Other Functions

There exist interesting equations that relate the gamma function to other special functions. These relationships are often quite surprising, because it is usually not immediately obvious that gamma function is related to other famous functions. The said famous functions I would like to explore now are the following:

1. Reimann Zeta Function

We will start with the following integral:

$$\int_0^{\infty}\frac{t^{x-1}}{e^t-1}dt$$

The first step is to multiply the integrand's numerator and denominator by e^{-t}.

$$\int_0^\infty \frac{t^{x-1}}{e^t - 1} dt = \int_0^\infty \frac{t^{x-1}e^{-t}}{1 - e^{-t}} dt$$

Using $\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k$, we can rewrite $\frac{1}{1-e^{-t}}$ as $\sum_{k=0}^{\infty} (e^{-t})^k$. Plugging this into the integral gives:

$$\int_0^\infty \frac{t^{x-1}e^{-t}}{1 - e^{-t}} dt = \int_0^\infty \sum_{k=0}^{\infty} e^{-kt} t^{x-1} e^{-t}\, dt$$

Once again, assuming that the order of summation and integration are interchangeable, we can rewrite the integral as:

$$\int_0^\infty \sum_{k=0}^{\infty} e^{-kt} t^{x-1} e^{-t}\, dt = \sum_{k=0}^{\infty} \int_0^\infty t^{x-1} e^{-(k+1)t} dt$$

$$Substitution: let\ u = (k+1)t, du = (k+1)dt$$

$$\sum_{k=0}^{\infty} \int_0^\infty t^{x-1} e^{-(k+1)t} dt = \sum_{k=0}^{\infty} \frac{1}{(k+1)^x} \int_0^\infty u^{x-1} e^{-u} du$$

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^x} = \sum_{k=1}^{\infty} \frac{1}{k^x} = \zeta(x), and\ \ \Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

$$\therefore \int_0^\infty \frac{t^{x-1}}{e^t - 1} dt = \zeta(x)\Gamma(x),$$

where $\zeta(x)$ is the Reimann zeta function. This relationship provides another alternative representation of the gamma function in terms of the Reimann zeta function as:

$$\Gamma(x) = \frac{1}{\zeta(x)} \int_0^\infty \frac{t^{x-1}}{e^t - 1} dt$$

Not only does this relationship provide a new way of expressing the gamma function, but it also enables us to find a general solution of the integral for different values of x. For instance, it is well known that $\zeta(2) = \frac{\pi^2}{6}$ and $\Gamma(2) = 1$. Using this we get:

$$1 = \frac{6}{\pi^2} \int_0^\infty \frac{t^{2-1}}{e^t - 1} dt$$

We can easily solve for the integral to obtain:

$$\int_0^\infty \frac{t}{e^t - 1} dt = \frac{\pi^2}{6}$$

2. Dirichlet Eta Function

A similar process can be carried out to obtain a similar equation that links the Gamma function with Dirichlet eta function. The result is the following:

$$\int_0^\infty \frac{t^{x-1}}{e^t + 1} dt = \eta(x)\Gamma(x),$$

where $\eta(x)$ is the Dirichlet eta function defined as $\eta(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x}$.

Using the fact that $\eta(x) = (1 - 2^{1-x})\zeta(x)$, we can rewrite this equation as:

$$\int_0^\infty \frac{t^{x-1}}{e^t + 1} dt = (1 - 2^{1-x})\zeta(x)\Gamma(x)$$

This equation can be used to evaluate integrals of the form $\int_0^\infty \frac{t^{x-1}}{e^t+1} dt$. For example, letting $x = 2$ yields:

$$\int_0^\infty \frac{t^{2-1}}{e^t + 1} dt = \eta(2)\Gamma(2) = (1 - 2^{1-2})\zeta(2)\Gamma(2)$$

$$\therefore \int_0^\infty \frac{t}{e^t + 1} dt = \frac{\pi^2}{12}$$

3. Beta Function (Euler Integral of the First Kind)

Beta function is a multivariable function defined in terms of the Gamma function. To derive it, we will start with the following product of two gamma functions.

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1} e^{-t} dt \int_0^\infty r^{y-1} e^{-r} dr = \iint_0^\infty t^{x-1}\, r^{y-1}\, e^{-(t+r)}\, drdt$$

Introducing the substitution $r = uv$ and $t = u(1 - v)$ requires the deduction of the new limits of integration, and $drdt$.

To determine the new limits of integration, we will start by noticing that r and t are both greater than 0. This means that u and v too must be greater than 0. However, v must also be less than 1, because otherwise t would be less than 0. To sum up: $u > 0$ and $0 < v < 1$.

To determine $drdt$, we have to evaluate the determinant of the Jacobian matrix.

$$\begin{vmatrix} \frac{\partial r}{\partial v} & \frac{\partial r}{\partial u} \\ \frac{\partial t}{\partial v} & \frac{\partial t}{\partial u} \end{vmatrix} = \begin{vmatrix} u & v \\ -u & 1-v \end{vmatrix} = u$$

Therefore $\Gamma(x)\Gamma(y)$ becomes:

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^1 u^{x+y-1}\, e^{-u}\, v^{y-1}\, (1-v)^{x-1} du dv$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty u^{x+y-1}\, e^{-u}\, du \int_0^1 v^{y-1}\, (1-v)^{x-1} dv$$

Using the integral definition of the gamma function, we can deduce that:

$$\Gamma(x)\Gamma(y) = \Gamma(\mathrm{x}+\mathrm{y}) \int_0^1 v^{y-1}\, (1-v)^{x-1} dv,$$

where $\int_0^1 v^{y-1}\, (1-v)^{x-1} dv$ is defined to be the Beta function, $\mathrm{B}(x,y)$.

We can therefore express the beta function in terms of the gamma function:

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Beta function also has an equivalent trigonometric form, which is extremely useful when evaluating trigonometric integrals. To derive it we will use the substitution $v = \sin^2\theta\,,\ dv = 2\sin\theta\cos\theta\, d\theta$.

$$\int_0^1 v^{y-1}\, (1-v)^{x-1} dv = \int_0^{\pi/2} \sin^{2y-2}\theta \cos^{2x-2}\theta\ 2\sin\theta\cos\theta\, d\theta$$

$$\therefore\ \mathrm{B}(x,y) = 2\int_0^{\pi/2} \sin^{2y-1}\theta \cos^{2x-1}\theta\, d\theta$$

Notice that the places of x and y can be switched without changing the result, due to the Beta function's symmetry.

To demonstrate the practicality of this form of the beta function, I would like to use it to solve an integral, which requires a relatively large amount of tedious work to be solved using standard methods of integration.

Consider:

$$\int_0^{\pi/2} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x\, dx$$

When written in this form, it is apparent that the integral is equal to $\frac{1}{2}\mathrm{B}\left(\frac{3}{4},\frac{1}{4}\right)$. Therefore, we can deduce that:

$$\int_0^{\pi/2} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x\, dx = \frac{1}{2}\mathrm{B}\left(\frac{3}{4},\frac{1}{4}\right) = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(\frac{3}{4}+\frac{1}{4})}$$

Using Euler's reflection formula:

$$\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(1-\frac{3}{4}\right) = \frac{\pi}{2}\csc\frac{3\pi}{4} = \frac{\pi\sqrt{2}}{2}$$

$$\therefore \int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi\sqrt{2}}{2}$$

4. Digamma Function

Digamma function is defined as the derivative of the natural logarithm of $\Gamma(x)$.

$$\psi(x) := \frac{d}{dx}\ln\Gamma(x)$$

Using the Weierstrass definition of the gamma function, we can express the reciprocal of the Gamma function as the following:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{k=1}^{\infty}\left(\frac{k+x}{k}\right)e^{-\frac{x}{k}}$$

Taking the natural logarithm of both sides gives:

$$\Rightarrow -\ln\Gamma(x) = \ln x + \gamma x + \sum_{k=1}^{\infty}\ln\left(\frac{k+x}{k}\right) - \frac{x}{k}$$

$$\Rightarrow \ln\Gamma(x) = -\ln x - \gamma x + \sum_{k=1}^{\infty}\frac{x}{k} - \ln\left(\frac{k+x}{k}\right)$$

Differentiating both sides gives the following expression for the Digamma function:

$$\psi(x) = -\frac{1}{x} - \gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+x}$$

Derivative of the Gamma function shows up frequently in different branches of mathematics. In fact, it is commonly represented in terms of the Digamma function to make its evaluation easier at particular values of x. This representation is as follows:

$$\Gamma'(x) = \Gamma(x)\psi(x)$$

Calculating the values of $\psi(x)$ at integer values of x is an interesting process, one that involves harmonic series, which we have previously discussed when deriving the Weierstrass representation of $\Gamma(x)$ (this is why I took the time to derive it).

To find a general expression for $\psi(n)$, where n is an integer, we will start by splitting the sum in the definition of $\psi(x)$ as follows:

$$\Rightarrow \psi(n) = -\frac{1}{n} - \gamma + \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+n} + \sum_{k=n+1}^{\infty} \frac{1}{k} - \frac{1}{k+n}$$

It becomes apparent that the sums are telescoping once we take a look at the partial sums:

$$\left[1 - \frac{1}{n+1} + \frac{1}{2} - \frac{1}{n+2} \pm \cdots + \frac{1}{n} - \frac{1}{2n}\right] + \left[\frac{1}{n+1} - \frac{1}{2n+1} + \frac{1}{n+2} - \frac{1}{2n+2} \pm \cdots\right]$$

The partial sums cancel out to give H_n (harmonic series), producing the following expression for $\psi(n)$:

$$\psi(n) = -\frac{1}{n} - \gamma + \sum_{k=1}^{n} \frac{1}{k}$$

Since the nth harmonic number is $\frac{1}{n}$, the final term in the sum can be taken out of the sigma notation to cancel with the $-\frac{1}{n}$ term, giving the following final equation:

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}, \qquad \textit{or equivalently}\text{: } \psi(n) = -\gamma + H_{n-1}$$

We can therefore express the derivative of the Gamma function at integer values as:

$$\Gamma'(n) = (-\gamma + H_{n-1})(\mathrm{n} - 1)!$$

For example, $\Gamma'(2)$ is equal to $(-\gamma + H_{2-1})(2-1)!$, which evaluates to $1-\gamma$, and $\Gamma'(3)$ is equal to $(-\gamma + H_{3-1})(3-1)!$, which evaluates to $3-2\gamma$.

Interesting Problems Involving the Gamma Function

The Gamma function and the other identities that we have derived so far come in handy when solving seemingly unrelated problems. Making use of the gamma function simplifies many problems and provides beautiful solutions. I would now like to work through a few problems to demonstrate this.

- **Problem 1**

$Evalueate\ the\ following\ generalised\ integral:$

$$\int_0^\infty \frac{dx}{x^n+1}$$

To solve this problem, we will make use of the Leibniz integral rule as we did before.

$$\Longrightarrow let\ I = \int_0^\infty \frac{dx}{x^n+1} \qquad let\ I(t) = \int_0^\infty \frac{1}{x^n+1} e^{-(x^n+1)t} dx$$

Introducing the parameter t in this way is a little counterintuitive, but it will make the calculation way easier.

$$\Longrightarrow \frac{dI(t)}{dt} = \int_0^\infty \frac{\partial}{\partial t}\left(\frac{1}{x^n+1} e^{-(x^n+1)t}\right) dx = -\int_0^\infty e^{-(x^n+1)t}\, dx$$

$$\Rightarrow \frac{dI(t)}{dt} = e^{-t} \int_0^\infty e^{-tx^n} dx$$

$$Substitution: let\ u = tx^n, \qquad du = tnx^{n-1}dx, \qquad dx = \frac{1}{n}\ t^{-\frac{1}{n}}\, u^{\frac{1}{n}-1} du$$

$$\frac{dI(t)}{dt} = -\frac{1}{n}\ t^{-\frac{1}{n}} e^{-t} \int_0^\infty u^{\frac{1}{n}-1} e^{-u} du$$

Since $\int_0^\infty u^{\frac{1}{n}-1} e^{-u} du = \Gamma\left(\frac{1}{n}\right)$, we get:

$$\frac{dI(t)}{dt} = -\frac{1}{n}\, t^{-\frac{1}{n}} e^{-t} \Gamma\left(\frac{1}{n}\right)$$

To get back to the goal integral, I, we will integrate $\frac{dI(t)}{dt}$ with respect to t from 0 to infinity. This gives the following result by the fundamental theorem of calculus:

$$I = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) \int_0^\infty t^{-\frac{1}{n}} e^{-t}\, dt$$

Since $\int_0^\infty t^{-\frac{1}{n}} e^{-t}\, dt = \Gamma\left(1 - \frac{1}{n}\right)$, we get:

$$I = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(1 - \frac{1}{n}\right)$$

Using Euler's reflection formula, we can conclude that:

$$I = \int_0^\infty \frac{dx}{x^n + 1} = \frac{\pi}{n}\csc\left(\frac{\pi}{n}\right)$$

We made use of the integral definition of the Gamma function, and Euler's reflection formula to arrive at this result, which is an interesting generalisation. Beta function could be used in an alternative solution to this integral, which also relies on the gamma function.

- **Problem 2**

Find the Laplace transform of the natural logarithm of $t, \mathcal{L}\{\ln t\}$.

$\mathcal{L}\{\ln t\}$ is defined as:

$$\mathcal{L}\{\ln t\} = \int_0^\infty \ln t\, e^{-st} dt$$

$$Substitution: let\ u = st, \qquad du = s dt$$

$$\mathcal{L}\{\ln t\} = \frac{1}{s}\int_0^\infty \ln\frac{u}{s} e^{-u} dt = \frac{1}{s}\left[\int_0^\infty \ln u\, e^{-u} du - \ln s \int_0^\infty e^{-u} du\right]$$

The hardest part of finding $\mathcal{L}\{\ln t\}$ is evaluating $\int_0^\infty \ln u\, e^{-u} du$, which cannot be solved using standard methods of integration. To solve this integral, we will first take a look at the derivative of the Gamma function. We will once again use differentiation under the integral sign.

$$\Gamma'(x) = \int_0^\infty \frac{\partial}{\partial x}(t^{x-1}\, e^{-t})dt = \int_0^\infty t^{x-1} \ln t\, e^{-t} dt$$

It can be seen that $\int_0^\infty \ln u\, e^{-u} du$ is equal to the derivative of the gamma function at $x = 1$. To evaluate $\Gamma'(1)$, we need to make use of the formula we derived using digamma function for $\Gamma'(x)$ at integer values:

$$\Gamma'(1) = (-\gamma + H_{n-1})(\mathrm{n} - 1)!$$

$$\therefore \Gamma'(1) = -\gamma$$

We can now conclude that:

$$\int_0^\infty \ln u\, e^{-u} du = -\gamma$$

This result finally allows us to find the Laplace transform of $\ln t$.

$$\mathcal{L}\{\ln t\} = \frac{1}{s}\left[-\gamma - \ln s \int_0^\infty e^{-u} du\right]$$

$$\therefore \mathcal{L}\{\ln t\} = -\frac{\gamma + \ln s}{s}$$

- **Problem 3**

Evaluate the Gaussian integral, $\int_{-\infty}^{\infty} e^{-x^2} dx$ *, using the gamma function.*

We can use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ to calculate the value of the Gaussian integral.

$$Gaussian\ Integral = \int_{-\infty}^{\infty} e^{-x^2} dx, \quad substitution: let\ x = \sqrt{t}, dx = \frac{dt}{2\sqrt{t}}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^\infty \frac{1}{2t^{\frac{1}{2}}} e^{-t} dt = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \Gamma\left(\frac{1}{2}\right)$$

$$\therefore Gaussian\ Integral = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

I believe this is a very elegant method of evaluating the Gaussian integral, because it does not rely on polar coordinates and double integrals.

Conclusion

The interesting problem of finding a generalisation of the factorial led to finding a function that is an extension of the factorial to real numbers, which is $\Gamma(x)$. We then found four different representations of this function, and used them to derive an interesting result: Euler's reflection formula, which relates a product of gamma functions to a trigonometric function. This alone is a very interesting finding, because the Gamma function and trigonometric functions do not even seem remotely related, yet we managed to connect them somehow.

We then derived two other functions that are closely related to the gamma function: the Beta function and the Digamma function, both of which proved useful when evaluating certain integrals. Furthermore, we found interesting relations of the gamma function to the Reimann zeta function and the Dirichlet eta function, which are both extremely important in analytic number theory. Given the fact that the million-dollar problem of the Reimann Hypothesis is all about the Reimann zeta function, I believe it is safe to say that the Gamma function might play a role in solving it.

We finally put our findings into practice by solving three problems. These problems demonstrated the practicality of the gamma function and the other results we have arrived at. We combined many other different tools from mathematics along the way, including the Leibniz integral rule, Laplace transform and Taylor series.

There are so many more applications of the gamma function, including gamma distribution from statistics and quantum physics, which I did not even begin to explain. The Gamma function's relations to other functions are not limited to the ones I demonstrated. It is connected to many other functions in many other ways. I also limited the gamma function's inputs to real numbers; however, it is possible to extend it to complex numbers as well. In short: the gamma function shows up constantly in mathematics. It was impossible for me to explain everything about it, so I picked my favourites to include.

Despite not including everything I had in mind, I believe I managed to emphasise the importance and beauty of extending a simple concept into a larger set of numbers using the Gamma function as a generalisation of the factorial. This is therefore one of my favourite concepts in mathematics.