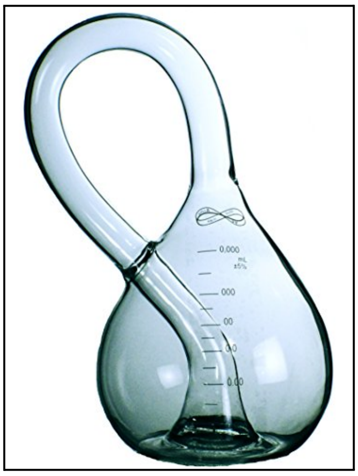
By Rajat Chitturu

Klein Bottles

I want you to imagine this. What happens when you stick two Möbius strips together? To find that out let's think about something much simpler. What happens when you stick two rectangles together? We get another rectangle, and end up with fewer ends than what we started with (eight edges turns to four edges). This tells us that when we stick two things together, we get something with fewer edges. A Möbius strip has only one edge. So when we stick two together, we must get something with less than 1 edge, which is 0 edges?!!?! This is what mathematicians call a Klein bottle, and it requires four spectral dimensions to exist. I am not going to explain what 4D actually means because, frankly, it is very confusing and I can barely get a grasp on it. All you need to know is that 4D is an extension of the three dimensional Euclidean space and gives a Klein bottle the property where it has only one surface and zero edges. The Klein bottle can be constructed (in a 4D space, because in a Euclidean space it cannot be done without allowing the surface to intersect itself) by joining the edges of two opposite-handed Möbius strips together. Opposite-handed Möbius strips are almost identical, but have a small difference, one strip will have a twist that goes over and the other strip will have a twist that goes under. Mathematicians distinguish between these two slightly different Möbius strips by calling them left and right handed Möbius strips.

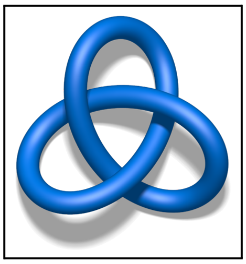
1) Proving A Klein Bottle has Only One Surface and Zero Edges

In 1882, the mathematician Felix Klein came up with the genius idea to take a cylinder, bring it through itself, and have the end of the cylinder welded to its base. This created an object that had only one surface: a non-orientable surface. To understand this, imagine an ant walking on the surface of a Klein bottle. They could continuously walk across its surface and never meet an edge making the inside and outside the same. It only has one side. They never crossed an edge to get to the inside. Let's compare this to an ordinary bottle. If you remove the cap of a bottle, you can clearly tell that there is an inside and an outside, which is separated by an edge, the lip of the bottle. If you make the bottle thinner and thinner, the lip of the bottle will become sharper. The ant wouldn't be able to get into the ordinary bottle without cutting themself. This shows that the ordinary bottle has two sides, the inside and the outside and is separated by the lip of the bottle; this is unlike a Klein bottle, which has only one surface.

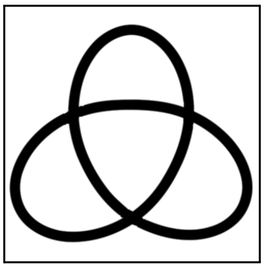


2) Things to Consider

You can not have a Klein bottle in real life because as I stated before, it can only exist in a 4D world. Since we live in a Euclidean space, the Klein bottles we use in real life are manifolds. A manifold is an object that closely resembles what the actual thing is. When we make a model, it is three dimensional because we live in three dimensions and can't really make four dimensional surfaces. You can see the problems this causes when you look at the Klein bottle manifold: the loop self-intersects the bottle. However, in a 4D space, there are no intersections. To understand why this happens, let's look at a Euclidean space object, more specifically a trefoil knot.



As you can see, the object has no self-intersections in a Euclidean space. But look at it in 2D:

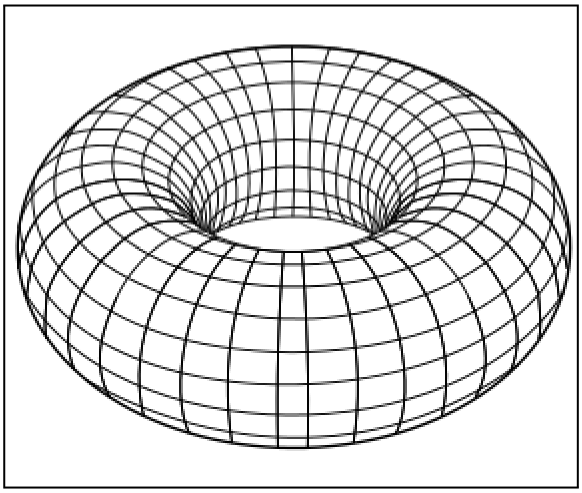


The object self-intersects 3 times. Similarly, a Klein bottle has 1 self intersection in the Euclidean plane, and it won't have any self intersections in a 4D plane. The self intersection is simply a figment of the object in a Euclidean space.

A Klein bottle is a two dimensional manifold (has only one surface) that can only exist in four spatial dimensions, similarly to a Möbius strip. However, a Klein bottle is a two dimensional compact manifold **without** a boundary (a Klein Bottle doesn't have an edge). As we learnt earlier with the ant analogy, the ant never encounters a barrier while moving on a Klein bottle. These properties make it a closed manifold.

3) Does a Klein Bottle have a Volume?

If you ask a mathematician whether a Klein bottle has a volume, they would say no. Let's look at other topological spaces, for example a torus.



A mathematician can definitely say that it does have volume. This is because it clearly separates the outside from the inside. A Klein bottle can be compared to a measuring cup, where it can **store volume**, but itself does not have a volume. The way topologists explain this is by showing that a measuring cylinder is homeomorphic to a disk. This means that a measuring cup will have the same properties as a flattened disk. The measuring cup can be melted down and flattened out to become a disk, which itself does not have a volume. It will be similar to a piece of paper which is technically a 3D object but the height is so small it is ignored and we consider it 2D. Similarly, topologists show that a Klein bottle doesn't have a mathematical volume. But does a Klein bottle have a physical volume, i.e. can it store volume?

The Klein bottles that we can make in a Euclidean space can store volume, similar to how measuring cups can. If a measuring cup is upside down, the liquid will be poured out due to gravity. Likewise, a Klein bottle can be filled with liquid and the liquid can be poured out. A Klein bottle works differently from a sphere that can store liquid, as some liquid in a Klein bottle can't be poured out.

4) What Happens When You Cut a Klein Bottle in Half?

We found out that if you put two Möbius strips together, you get a Klein bottle. Does this work in the other direction? Will you get two Möbius strips when you cut a Klein bottle in half? To test this out, we can use a fabric Klein bottle that has a zipper in the middle, instead of a glass/plastic one.

When you unzip the Klein bottle, you indeed get two opposite handed Möbius strips.

But what happens when you cut a glass/plastic Klein bottle in half? When you cut a glass/plastic Klein bottle in half you get this:

To test if these are two Möbius strips, let's see if it has one edge. Remember that in a Euclidean space, a Klein bottle self-intersects itself, because it requires four spatial dimensions to exist. This means that when the Klein bottle is cut in half, the edge won't continue due to the self intersection in a Euclidean space, but will continue in four dimensions. Now that we have understood that, we can simply follow the edge and see that the object has one edge. Similarly, you can do this by seeing if the object has only one surface, which it does. This proves that you do, in fact, get two Möbius strips when you cut a glass Klein bottle in half, one left handed and one right handed.

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