Kepler’s Laws and Anomalies of the Universe

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Astronomy is one of the oldest disciplines of natural science for exploring the motion of the planets, the solar system and beyond. How was the solar system created? How old is the sun? Or how many stars are in the Milky Way? These are just some questions among many that astronomers can answer.

The position, velocities and mass of astrophysical objects can be found using one the most prominent and old branches of mathematics; geometry. Or rather, Astrometry. A very important branch in astronomy is the study of planetary motion, from the days of Aristotle suggesting the geocentric model with the Earth at the center, to Copernicus and Galileo advocating for the heliocentric model with the Sun at the center.

The subject was revolutionized by German mathematician and astronomer Johannes Kepler as he introduced three laws in the 17th century which governed the motion of all planetary systems. We would like to discuss briefly his findings and their numerous applications and how they lead to other fascinating findings. It should be noted that the theorems are stated with respect to the Sun but these laws apply for any orbital motion in space.

The three laws are as follows: the law of orbits, the law of equal areas, and the law of periods.

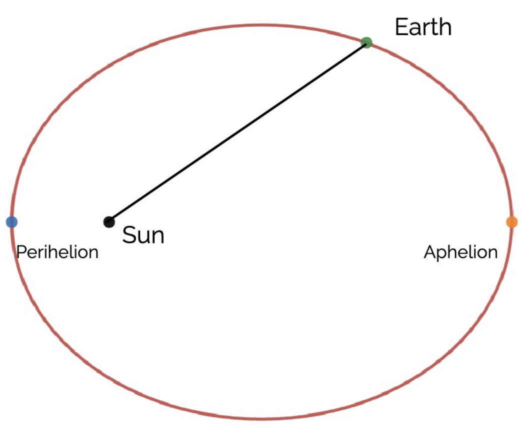


Figure 1

Kepler’s first law: Each planet's orbit about the Sun is an ellipse with the Sun at a focus.

Not only does this allow us to predict trajectories and positions of planets and other moving bodies, it disproved a common belief of a divine symmetry of the planets, which even Kepler believed, which stated that orbits were circular.

Although Kepler came across this by examining numerous tables and statistical data, for our purposes we can derive the laws from Newton’s Universal law of gravitation, taken from Principia.

We will not delve fully into the details of the first proof but we will give an introduction and show how it might lead to some interesting properties.

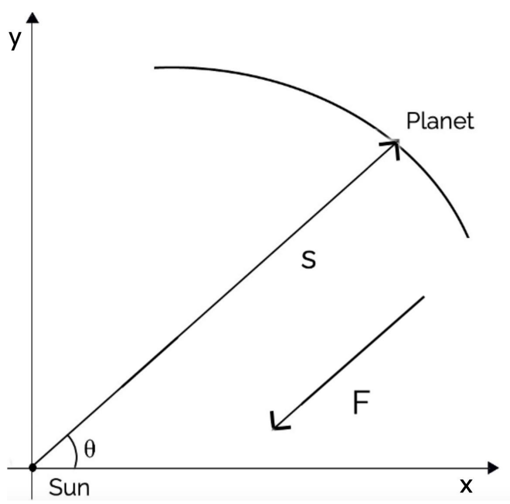
Newton's Universal Law of Gravitation states that every particle attracts every other particle in the universe with force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Simply put, let us use our example of the Sun and an arbitrary planet. Let the Sun have mass **M**, and the planet have mass **m**. Now the law uses the distance between the masses, say **r**. This suggests that one should use polar coordinates for convenience. The constant of proportionality here is **G**, a known physical constant. The force **F**, directed towards the sun can then be written as

$$F = G\frac{Mm}{r^2}$$

(1)

If we look at the following diagram with the following notation



$$\theta = angle\ made\ with\ x - axis$$
$$s = position\ vector\ of\ planet$$
$$r = distance\ function\ of\ planet$$
$$a = acceleration\ of\ planet$$

Figure 2

It is important to note here that the Sun is taken as the origin.

From these it follows that

$$s = r cos\theta\, i + r sin\,\theta\, j$$

(2)

Hence the unit vector in the direction of **s** can be written as

$$s_r = cos\theta i + sin\theta j$$

(3)

Observing the direction of the force, as it is the Sun that is attracting the planet, we can redefine the force **F** by writing it as a vector in the direction opposite of **s** with length

$$G\frac{Mm}{r^2}$$

Hence, we can write in combination with Newton's second law of motion

$$\vec{F} = -G\frac{Mm}{r^2}s_r = ma$$

This means

$$a = -GMs_r$$

(4)

From this part of the proof we can make an interesting observation. The acceleration vector is always parallel to the position vector of the planet and is directed towards the opposite direction. This means that the acceleration is radial.

Using Kepler's first law, we can also make observations about where perihelion (point closest to the sun) and aphelion (point farthest from the sun) lie. Now we should mention some facts relating to ellipses. An ellipse is a conic section. This means for every ellipse there exists a fixed point (called a focus) and a fixed line (called the directrix) such that the ratio of the distance from an arbitrary point on the ellipse to the focus and the perpendicular distance of that point to the directrix is constant. This constant is known as the eccentricity of the ellipse. Given an ellipse of eccentricity **e** and distance from the focus to directrix **d**, the polar equation of the ellipse is given by the following equation. We also have 0 < **e** < 1.

$$r = \frac{ed}{1 + ecos\theta}$$

(5)

We can easily find the largest and smallest values of **r**, they are given correspondingly

$$r = \frac{ed}{1-e} \qquad\qquad r = \frac{ed}{1+e}$$

So we can see that there is perihelion when **θ** = 0 and aphelion when **θ** = π.

Note that in the diagram at the beginning, the diagram is rotated by π radian hence the values for **θ** are switched. Using the property of radial acceleration, we can deduce Kepler's second law which we state now.

Kepler's second law: The radius vector drawn from the Sun to the planet sweeps out equal areas in equal intervals of time.

To show this visually, let us consider any three separate instants $\mathbf{t_1}$,$\mathbf{t_2}$ and $\mathbf{t_3}$. Kepler's second law says that the area of the region of the ellipse from $\mathbf{t_1}$ to $\mathbf{t_1}$+$\mathbf{t_2}$ and the area of the region of the ellipse from $\mathbf{t_3}$ to $\mathbf{t_3}$ + $\mathbf{t_2}$ will be equal.

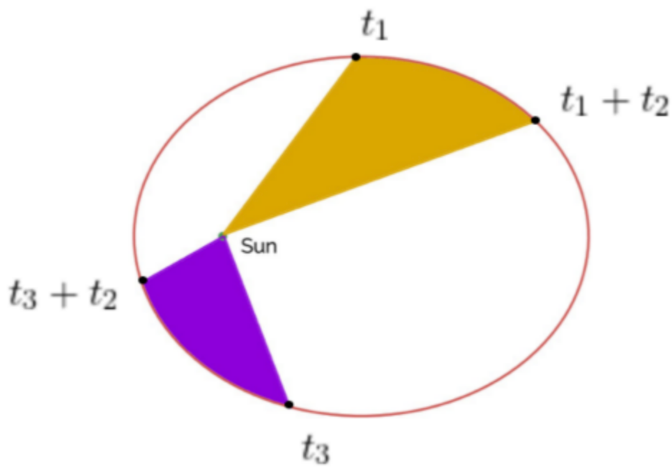


Figure 3

This statement is equivalent to saying that the derivative of the Area function of the ellipse (the function which gives us the area of a certain region) is constant. To prove this we will use a theorem which allows us to compute area in polar coordinates. The theorem is stated as follows:
Let **r** = **f(θ)** be the equation of the polar function **r**, continuous on an interval [**α**,**β**], where **β - α** ≤ 2π. Then the area of the region bounded by **f(θ)** as **θ** varies over the interval [**α**,**β**] is given by the formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)d\theta$$

Now we cannot directly apply this formula to (2), as **r** and **θ** are both functions of time. Let **r** = **f(t)** and **θ** = **g(t)**.

Assume **r** can be written as a function of **θ**, say **r** = **R(θ)**. We wish to find the area, say A, over the interval [**a**,**t**] for some **a** as time **t** varies. Then the formula for area becomes

$$A = \frac{1}{2}\int_{g(a)}^{g(t)} R^2(\theta)d\theta$$

(6)
Differentiating both sides, using the chain rule and the first fundamental theorem of calculus we have

$$\frac{dA}{dt} = \frac{1}{2}\frac{dg}{dt}R^2(g(t))$$

Substituting **g(t)** by **θ** and substituting **R(θ)** by **r** we have

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

So we have to show that this is constant.
Consider the following functions

$$s_\theta = -sin\,\theta i + cos\,\theta j \qquad v = \frac{ds}{dt} \qquad a = \frac{dv}{dt}$$

In conjunction with (2) and (3) we have

$$s = rs_r$$

(Differentiating both sides with respect to t)

$$v = \frac{dr}{dt}s_r + r\frac{d\theta}{dt}s_\theta$$

(Differentiating both sides with respect to t)

$$a = (\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2)s_r + (r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt})s_\theta$$

It is worth noticing that $\mathbf{s_r}$ is orthogonal to $\mathbf{s_\theta}$ and both have length 1 hence they are linearly independent. Comparing this with (4) we have

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

Or, multiplying with **r**, we have

$$r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

Hence

$$(r^2\frac{d\theta}{dt})' = 0$$

Thus proving that $\frac{dA}{dt}$ is constant, proving Kepler's second law.

Kepler's third law: The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axes of their orbits.

Let **T** be the time taken to complete one orbit, and let **p** be the length of the semi-major axis and let **b** be the length of the semi-minor axis.

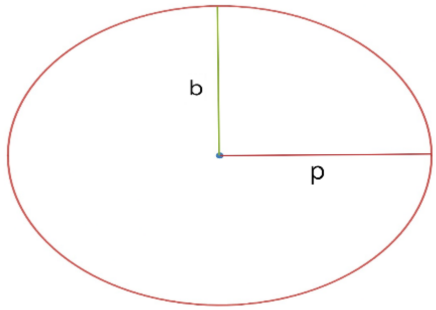


Figure 4

Kepler's third law states that

$$T^2 = \frac{4\pi^2}{GM}p^3$$

(7)

We derive this from the following equations.

$$\begin{aligned}
&(a)\ \ Area\ of\ ellipse = \pi pb \\
&(b)\qquad \frac{1}{2}kT = \pi pb \\
&(c)\qquad b^2 = p^2(1-e^2) \\
&(d)\qquad k^2 = GMp(1-e^2)
\end{aligned}$$

Where **k** is the constant rate of increase in area shown in Kepler's second law.

Equation (c) and Equation (d) use the following formulas, which can all be derived from the definition of an ellipse and Kepler's first law, but we will assume it is true for our purposes.

$$r = \frac{ed}{1+ecos\theta} \qquad p = \frac{ed}{1-e^2} \qquad d = \frac{k^2}{GMe}$$

This property has some interesting consequences. It shows that planets with smaller orbits have faster periods. For example, Mercury takes 88 days to orbit the sun, the Earth takes 365 days, and Saturn takes 10,759 days.

Another property we find is that in any orbit, the time taken for one complete rotation is dependent only on the mass of the central object and the length of the semi-major axis.

If a planet has moons, their orbital period can be used to determine the mass of the planet. Jupiter has the maximum number of moons among all the planets in the solar system, thus an early measurement of its mass was possible due to this reason.

We now give a few remarks about the orbital speed of planets.

We use the law of conservation of energy that states that energy is neither created or destroyed and total energy is conserved at any point in the orbit. If we let **v** denote the speed function of a planet at each point in the orbit and **m** denote its mass, we have the following. The position and the force are functions of time.

$$Total\ energy = Kinetic\ energy + Potential\ energy$$

$$Kinetic\ energy = \frac{1}{2}mv^2$$

$$F = -\frac{d}{dt}(Potential\ energy)$$

Using (1)

We have

$$Potential\ energy = -\frac{GMm}{r}$$

$$Total\ energy = -\frac{GMm}{r} + \frac{1}{2}mv^2$$

(8)

Since this total energy is constant, we notice that when the planet is at perihelion, **r** is at its minimum, and hence the speed is at its maximum, meaning it is at its fastest. Similarly we see that the speed is at its minimum at aphelion, meaning it is at its slowest.

Violations: Even after proving each law, it seems that there is more to be found. Kepler's first law stated that planets move in an ellipse with the Sun at a foci, and it only makes sense that the position of perihelion be constant.

But this is not always the case. Planetary orbits are not always perfect ellipses. This can be due to perturbations produced by other neighboring objects with significant mass and gravitational pull. This led French astronomer Urbain Le Verrier and British mathematician John Couch Adams to the discovery of Neptune independently by observing perturbation in the orbit of Uranus.

It is also noticed that Mercury's perihelion moves every year by a small angle. Even after accounting for external gravitational force, there was an unaccounted shift in the angular position of the perihelion of 43 arc-seconds (0.01°) per century. This shift can be explained with the theory of general relativity, according to which all orbits are not closed ellipses.

Existence of Dark Matter

If we observe a galaxy under the assumption that stars move in circular orbits or in orbits with low eccentricity, we have centripetal acceleration **a**, or, we have the equation given below. (Note there is uniform circular motion due to Kepler's second law).

$$a = \frac{v^2}{r}$$
(9)

Combining this with the Universal Law of Gravitation and Newton's second law of motion we have

$$\frac{GMm}{r^2} = ma = \frac{mv^2}{r}$$
$$v^2 = \frac{GM}{r}$$
$$v = \sqrt{\frac{GM}{r}}$$
(10)

Where **v** is the speed, **r** is the distance from the center, **m** is the mass of the star and **M** is the mass of the center.

We have that the speed is inversely proportional to the square root of the distance, hence as we consider stars further from the galaxy's center, we expect that speed will continue to decrease and will get very close to zero.

However, this is not the case when we assume the homogeneous mass density of the galaxy is somewhat constant, say ρ.

We know from Kepler's third law and (7) that the mass **M** should be proportional to **r** cubed. Say

$$M = M_0 r^3$$

Putting this into the equation we find

$$v = r\sqrt{GM_0}$$
(11)

So the speed is directly proportional to the distance in the dense part of the galaxy, closer to the center. Combining the above expressions of **v**, we plot **v** against **r** and get what is known as a galaxy rotation curve. The cusp occurs more gradually as the density of the galaxy changes.

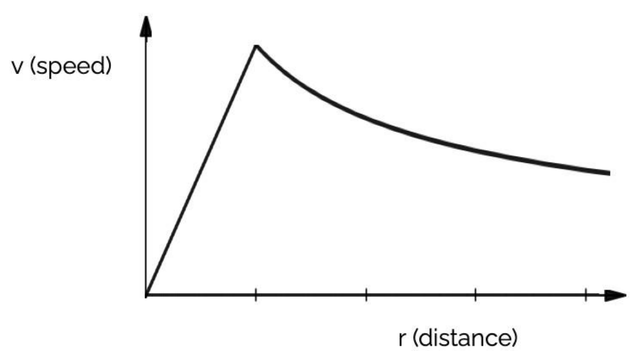


Figure 5

But if we plot the observational data we see a different trend.

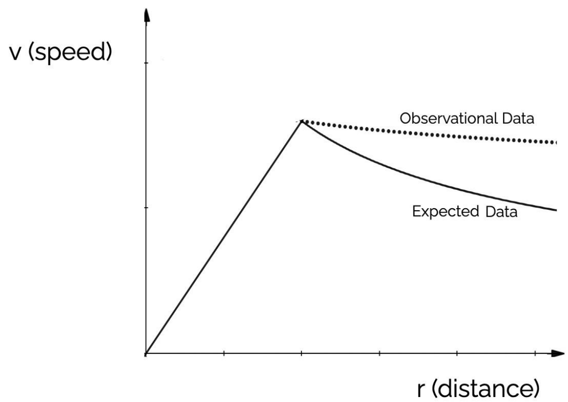


Figure 6

This phenomenon occurs even in the Milky Way. This shows that there may be more matter in galaxies which we do not know about. This is called ‘dark matter’.

Concluding Remarks

The above examples show that when comparing expectations to reality, we find outliers, and these outliers can be the milestones to fascinating discoveries. This happens often and can lead scientists to think differently and reveal more about the universe.

Even though we used Newton’s law of gravitation as a vital tool in deriving Kepler’s laws, in reality it is worth noting that Kepler published them in 1609, whereas Newton published the law of gravitation in 1687, influenced by Kepler’s discoveries.

It is easy to take these laws for granted but these laws uncover an incredible amount about the universe and motion of planets and moving bodies in general. For being more precise in Astrometry, one would need to apply the corrections derived using general relativity, but these laws alone will get you very far.

References

Apostol, Tom. Calculus, Volume 1. Second Edition, John Wiley & Sons, Inc., 1967.

Applying Kepler's Laws. *Brilliant.org*. Retrieved 21:01, March 4, 2023, from https://brilliant.org/wiki/applying-keplers-laws/

Justin Babcock, Secondary Science Teacher. Rotational Curves and the Mass of Galaxies. 2020. https://youtu.be/MN1aexJ18o8

Kepler’s Laws of Planetary Motion. BYJU’s. 2022. https://byjus.com/jee/keplers-laws/

Zhabinskaya, Dina. 2.5: Force and Potential Energy. 2019.
https://phys.libretexts.org/Under_Construction/Purgatory/2%3A_Applying_Models_to_Mechanical_Phenomena/2.5%3A_Force_and_Potential_Energy#:~:text=and%20potential%20energy.-,Force%20and%20Potential%20Energy,the%20function%20at%20some%20point.

A.Kaviyarasu. History of Planetary Motion.
https://www.mitindia.edu/images/pdf/avionics_ppt/history%20of%20planetry%20motion.pdf

Davis, Joel. Astronomy.com, Who discovered Neptune? 2020.
https://astronomy.com/news/2020/11/who-discovered-Neptune

Alan Gould, Brian Dunbar. Kepler and K2. NASA. 2017. https://www.nasa.gov/kepler/education/johannes

Gee, Laura. "Who Was the First Person to Discover Gravity?" sciencing.com,
https://sciencing.com/first-person-discover-gravity-23003.html. 4 March 2023.