To Infinity and Beyond

What is infinity and how has it changed throughout history?

ABSTRACT

Infinity was a consequence of the curious and a question of the rational: it is a concept so fundamentally involved in mathematics and, yet, mathematicians have avoided formally defining it for centuries. What happens if we keep counting forever? How many numbers exist? Can we quantify the amount of numbers we can create? What happens if we want to create more numbers? Such questions have plagued the minds of the mathematicians and logicians, where the earliest scholars have historically described the "horrors" of infinity - perhaps it is an indication of the limits of mathematics representing the physical world.

1 A Concept

Infinity - a description of the limitless, boundless and uncountable - was an inevitable concept. It forced itself upon the Greeks through the physical qualities of the world: time seemed endless, space and time could be divided without limit and space was boundless. Before this, the Pythagoreans had vehemently denied its existence, good and evil were finite and infinite; the unmeasurable was unfathomable. At the time, it was not well understood that the concept of infinity was necessary for the Pythagorean discovery of irrational numbers such as $\sqrt{2}$ and the never-ending expansions of their decimal counterparts. The symbol, ∞, which dictates an unending curve was given by Wallis in the 1600s when infinity was properly formalised as a concept but not yet fully understood.[1]

Greek mathematicians had devised methods to supposedly work around infinity when messing planes and solid figures where an infinite "limit" process was used, known as the method of exhaustion: the main idea was to replace an infinite approximation by a double reductio ad absurdum (reduction to absurdity). For example, by labelling the area, a, of an arbitrary shape, s, it could be proved that $a = s$ by showing that $a > s$ and $a < s$ caused contradictions.[1] This ultimately led to the theory of indivisibles and points at infinity which emerged much later in the 1700s. Two notable mathematicians, Newton and Leibniz, would thus develop calculus which exploited limits and the infinite. Aristotle had previously defined minimal (potential) infinity, which was enough to facilitate theorems that were discovered without introducing a new number and yet avoided the actuality of infinity. Potential infinity is not a completed or "true" infinity: it refers to the idea of a finite, indefinitely large value. Meanwhile actual infinity is found with the regard of the totality of numbers itself as a completed unity. It was Cantor in the late 1800s and early 1900s who eventually developed the concept of the actual infinite.[7]

---

[1] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, 2. *Infinity in Mathematics: A Historical Overview*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023

[2] S.Singh, *Fermat's Last Theorem*, 1997, pages 84 - 85

[3] Hilbert's Paradox of the Grand Hotel, *Prime Factorisation Method*, Available at: https://en.wikipedia.org/wiki/Hilbert%27s_paradox_of_the_Grand_Hotel Date accessed: 12/02/2023

[4] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, *3.3 Infinities of Counting*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023

[5] G. D. Allen, Texas A&M University, *The History of Infinity*, pages 1 - 14

[6] Stanford University, Stanford Encyclopaedia of Philosophy, *The Continuum Hypothesis*, Available at: https://plato.stanford.edu/entries/continuum-hypothesis/, 2013, Date accessed: 12/02/2023

[7] D. Hilbert, Cambridge University, Philosophy of Mathematics, *On the Infinite*, Available at: https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html, Date accessed: 12/02/2023

2 Hilbert's Hotel

David Hilbert, a German mathematician in the 19th and early 20th century, defined infinity as a never-ending set of natural (counting) numbers and anything comparable in size was therefore also infinite. In 1924, Hilbert presented a paradox: Hilbert's hypothetical hotel has the desirable property of having an infinite number of rooms in which each room is occupied. One day, a new guest appears, requesting a room and is soon assured that a vacant room is available. The solution is intuitively simple: the clerk asks the guests currently occupying the room, n, to move to the room, $n + 1$, for example, the guests in room 1 are told to move to the next room, 2. This allows for room 1 to become available for this new guest. Conversely, this also applies if the guest in each thousandth room were to be moved to the next thousand and, despite this, the same amount of people will still be moved. This seemingly contradictory notion demonstrates that infinity plus one is infinity - a key property of infinity.

Consider another case: a countably infinite number of guests arrive on a train at Hilbert's Hotel, each requesting a room of their own, and so another problem arises: what is infinity plus infinity? The solution was, again, relatively simple: each guest in the nth room was told to move to the $2n$th room. This results in all the current guests occupying the even numbered rooms while the new guests were to occupy the odd. A few conclusions can be drawn from this, the first being that infinity plus infinity is, in fact, infinity and the second being that there was an equally infinite number of odd and even numbers where every even number can be paired with an odd number. This shows that the two infinities are the same in size, however other infinite quantities can be proven to be bigger than others.[2]

Finally, a train with infinitely many carriages arrives at the hotel with countably infinitely many passengers in each carriage, all requesting a room at a fully-booked hotel, still with infinitely many rooms. Unlike the previous two examples, the solution and consequences for this problem are strange. One solution in particular, known as the prime powers method, is as follows: each of the current guests occupying the nth room in the hotel are told to move to the 2^nth room and, since Euclid had proven that there were infinitely many primes in 300 BC, each of the carriages are assigned a different prime number, p, that is not 2. Each passenger in seat number, c, will then be assigned the room p^cth room, allowing for all new and current guests to have a room which will never overlap (a property of prime powers). This, however, causes a weird problem to arise: room numbers which are not prime powers, such as 12, 15 or 284 will become empty, suggesting that adding infinity to infinity an infinite number of times is less than infinity.[3] So how is this possible? Take the number of hotel rooms to be the set of natural numbers, and the number of carriages to be the set of prime numbers. Both of these sets have the same cardinality where each set is considered countably infinite (meaning each element of the set having a one to one correspondence with the set of natural numbers). Although these sets are equal in size, due to the fact that, for a given range, there are more natural numbers than prime, the set of prime numbers is much less dense than that of natural numbers. This provides a rather simple explanation for this paradoxical

---

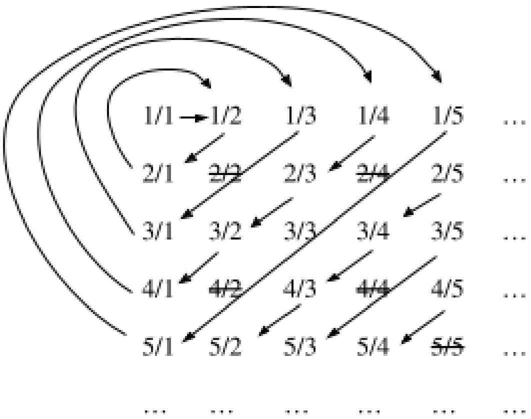
[1] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, 2. *Infinity in Mathematics: A Historical Overview*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023
[2] S.Singh, *Fermat's Last Theorem*, 1997, pages 84 - 85
[3] Hilbert's Paradox of the Grand Hotel, *Prime Factorisation Method*, Available at: https://en.wikipedia.org/wiki/Hilbert%27s_paradox_of_the_Grand_Hotel Date accessed: 12/02/2023
[4] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, *3.3 Infinities of Counting*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023
[5] G. D. Allen, Texas A&M University, *The History of Infinity*, pages 1 - 14
[6] Stanford University, Stanford Encyclopaedia of Philosophy, *The Continuum Hypothesis*, Available at: https://plato.stanford.edu/entries/continuum-hypothesis/, 2013, Date accessed: 12/02/2023
[7] D. Hilbert, Cambridge University, Philosophy of Mathematics, *On the Infinite*, Available at: https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html, Date accessed: 12/02/2023

problem: it is not that adding infinity to infinity infinitely is less than infinity, rather the result is instead a less "dense" infinity since its actual size does not decrease.

3 Set Theory

In 1932, Georg Cantor proposed the modern theory of countable infinities and the different "sizes" of infinity that existed. Countable infinity (of a set) can be defined as having a one-to-one correspondence to the set of natural numbers, often given the symbol, $\aleph_0$, as previously described by Hilbert. By considering two finite sets, Cantor observed that each element in such a set could be paired with an element from a second set which could also be applied to some infinite sets. For example the set of perfect squares and the set of natural numbers (the positive roots of the squares) have a one to one correspondence where each square can be directly associated with its root, so the set of perfect squares is countably infinite. In simpler terms, a set is countable if all the elements within the set could be listed, thereby having that correspondence that Cantor described.

Cantor's most intriguing early observations is that of the positive rational numbers. He could prove that every rational number - written in the form $\frac{m}{n}$ where *m* and *n* are positive integers and coprime (no common factors) - had an associated natural number. A visual proof, called Cantor's diagonal method, is as follows:



By writing the numbers systematically in a two dimensional array, instead of a linear one-dimensional list, and removing the rationals that simplify to make duplicates, the numbers could be ordered in such a way to eventually include every rational number and therefore a one-to-one correspondence with the natural numbers can be seen where each arrow indicates pairings of numbers.[4]

---

[1] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, 2. *Infinity in Mathematics: A Historical Overview*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023
[2] S.Singh, *Fermat's Last Theorem*, 1997, pages 84 - 85
[3] Hilbert's Paradox of the Grand Hotel, *Prime Factorisation Method*, Available at: https://en.wikipedia.org/wiki/Hilbert%27s_paradox_of_the_Grand_Hotel Date accessed: 12/02/2023
[4] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, *3.3 Infinities of Counting*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023
[5] G. D. Allen, Texas A&M University, *The History of Infinity*, pages 1 - 14
[6] Stanford University, Stanford Encyclopaedia of Philosophy, *The Continuum Hypothesis*, Available at: https://plato.stanford.edu/entries/continuum-hypothesis/, 2013, Date accessed: 12/02/2023
[7] D. Hilbert, Cambridge University, Philosophy of Mathematics, *On the Infinite*, Available at: https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html, Date accessed: 12/02/2023

Cantor also described infinite sets where there was no one-to-one correspondence, known as uncountable sets: the set of real numbers and the set of natural numbers. Intuitively, the set of real numbers is infinitely larger than that of the natural numbers, implying the notion that there exist higher infinities where $\aleph_0$ is the smallest infinity. This can, in fact, be proven using a method known as Cantor's Diagonal argument. Firstly, restrict to the subset of reals in the interval (0, 1) and consider the following 4 arbitrarily chosen numbers that are within the set:

$$a_1 = 0.10012294\ldots$$

$$a_2 = 0.12134500\ldots$$

$$a_3 = 0.41424566\ldots$$

$$a_4 = 0.77802013\ldots$$

A new number can be created by taking and collating the diagonal digits; in this case, a new number, 0.1240… is created. Then, by changing each digit to be an arbitrary digit that is not equal to its current digit in that position (usually by following a chosen rule such as checking if the current digit is 1 and replacing the 1 with 2 or, else, replace the digit with a 1), thus creating a new number that is not in the initial subset. This new number is not equal to the first element in the list as the first digit would be different, the new number is also not equal to the second as the second digits are distinct and so on, therefore proving that there is no one-to-one correspondence between the set of reals and the set of naturals, demonstrating that there are indeed higher infinities.[5]

4 The Continuum Hypothesis

The Continuum Hypothesis arises from Cantor's work on infinities, with set theory, appearing in 1877, and states that there exists no set that is both larger than the set of natural numbers while being less than the set of real numbers. Using $\aleph_0$ to denote the cardinality of natural numbers, the continuum hypothesis is written as $2 \wedge \aleph_0 = \aleph_1$, where $\aleph_1$ is the "greater" infinity, however, this hypothesis was incredibly difficult to prove. To this day, it is one of the most prominent open problems that exist in set theory, where the roots of the problem lie in the concept of the infinite, and as the problem persisted, Hilbert placed it on his infamous list of open problems to be faced in the 20th century.[6] It was only a few years later, in 1931, that Kurt Gödel's Incompleteness Theorem would provide an explanation as to why the hypothesis could not be proved or disproved with the current axioms in mathematics.

Conclusion

Infinity is used to describe things with no bound, no limit and no end, and yet the study of infinity in mathematics and philosophy has shown that it may not be so simple. From its strange paradoxical properties to

---

[1] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, 2. *Infinity in Mathematics: A Historical Overview*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023

[2] S.Singh, *Fermat's Last Theorem*, 1997, pages 84 - 85

[3] Hilbert's Paradox of the Grand Hotel, *Prime Factorisation Method*, Available at: https://en.wikipedia.org/wiki/Hilbert%27s_paradox_of_the_Grand_Hotel Date accessed: 12/02/2023

[4] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, *3.3 Infinities of Counting*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023

[5] G. D. Allen, Texas A&M University, *The History of Infinity*, pages 1 - 14

[6] Stanford University, Stanford Encyclopaedia of Philosophy, *The Continuum Hypothesis*, Available at: https://plato.stanford.edu/entries/continuum-hypothesis/, 2013, Date accessed: 12/02/2023

[7] D. Hilbert, Cambridge University, Philosophy of Mathematics, *On the Infinite*, Available at: https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html, Date accessed: 12/02/2023

its role in set theory, the principle of infinity is still shrouded in questions, perhaps eventually being resolved to satisfy logicians and mathematicians, however, for most people, a perfectly working and reasonable definition for infinity has already been achieved.

---

[1] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, 2. *Infinity in Mathematics: A Historical Overview*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023
[2] S.Singh, *Fermat's Last Theorem*, 1997, pages 84 - 85
[3] Hilbert's Paradox of the Grand Hotel, *Prime Factorisation Method*, Available at: https://en.wikipedia.org/wiki/Hilbert%27s_paradox_of_the_Grand_Hotel Date accessed: 12/02/2023
[4] Stanford University, Stanford Encyclopaedia of Philosophy, *Infinity*, *3.3 Infinities of Counting*, Available at: https://plato.stanford.edu/entries/infinity/#toc, 2021, Date accessed: 12/02/2023
[5] G. D. Allen, Texas A&M University, *The History of Infinity*, pages 1 - 14
[6] Stanford University, Stanford Encyclopaedia of Philosophy, *The Continuum Hypothesis*, Available at: https://plato.stanford.edu/entries/continuum-hypothesis/, 2013, Date accessed: 12/02/2023
[7] D. Hilbert, Cambridge University, Philosophy of Mathematics, *On the Infinite*, Available at: https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html, Date accessed: 12/02/2023