Unveiling Hidden Sounds: Beat Frequency and Fourier Transform

By Carys Timpson

At the start of an orchestral performance, there is a low hum and murmuring of notes as the musicians all tune their instruments to their 1st oboist. Do all the musicians have perfect pitch? Can they all detect exactly 440 Hz, the note 'A', on their various cellos, clarinets, flutes? The answer is no. So how do they know if they are in tune? The answer is beat frequency. Let's investigate the maths and physics behind it, link it to Fourier transforms and investigate the large range of its applications all around us.

Superposition

Sound is a wave that transfers energy from one place to another without transferring mass. Sound can only be transferred and heard through a medium that contains particles, so it cannot travel through a vacuum. Particles in the medium (in our case, air) vibrate parallel to the direction of energy transfer to create compressions and rarefactions which can be described as a longitudinal wave. A compression is a region of space where the particles are pulled together tightly creating a high-pressure region and conversely rarefactions are low-pressure regions.

The frequency of a sound wave is how many cycles of compressions and rarefactions there are in one second measured in Hertz; this determines the pitch of the note we hear. The higher the frequency, the higher the pitch. The middle C has a frequency of 261 Hz (concert pitch standard) and the note A that an orchestra tunes to has a frequency of 440 Hz. It is interesting to see the mathematical relationships between notes and their frequency. Did you know that if you play an octave higher than middle C, the frequency is exactly double at 522 Hz? All the notes in scales have mathematical relationships as well and some of these relationships sound more pleasing to the ear than others such as a chord made of a fifth. This is due to the fifth note being 1.5 times the frequency of the lower note.

The amplitude of a wave, the maximum displacement from the equilibrium position, is related to the energy of a wave and in the case of sound waves, how loud the sound will be to our ears.

Sound waves can be modeled as sinusoidal graphs with the generalised equation:

$$x = A\sin(2\pi f t)$$

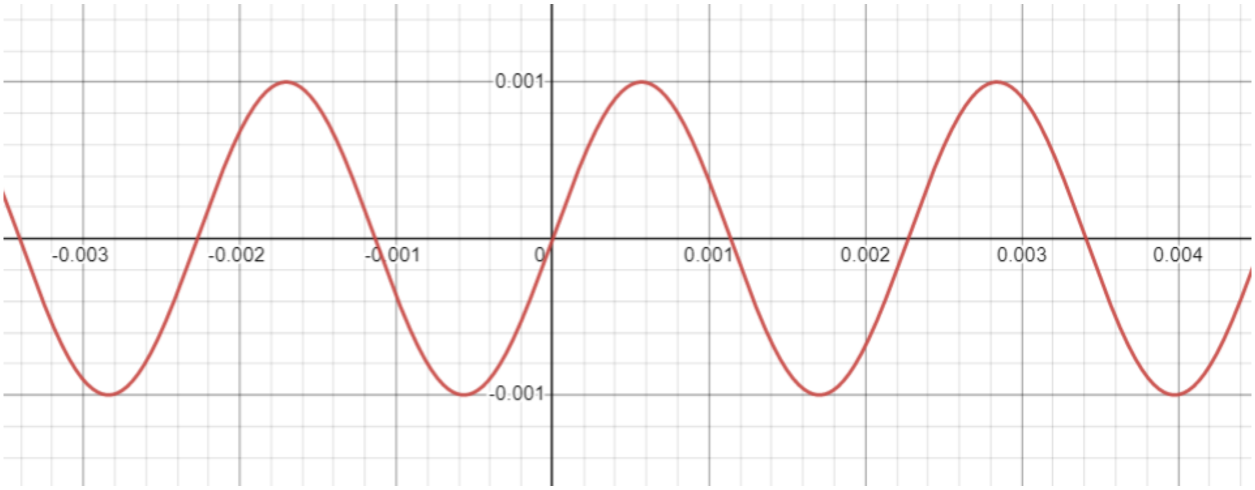
A = amplitude of the wave and therefore related to how loud the sound

f = frequency of the wave in Hertz

Here is a graph of a sound wave of 440 Hz where the peaks represent the compressions and the trough represent the rarefactions.

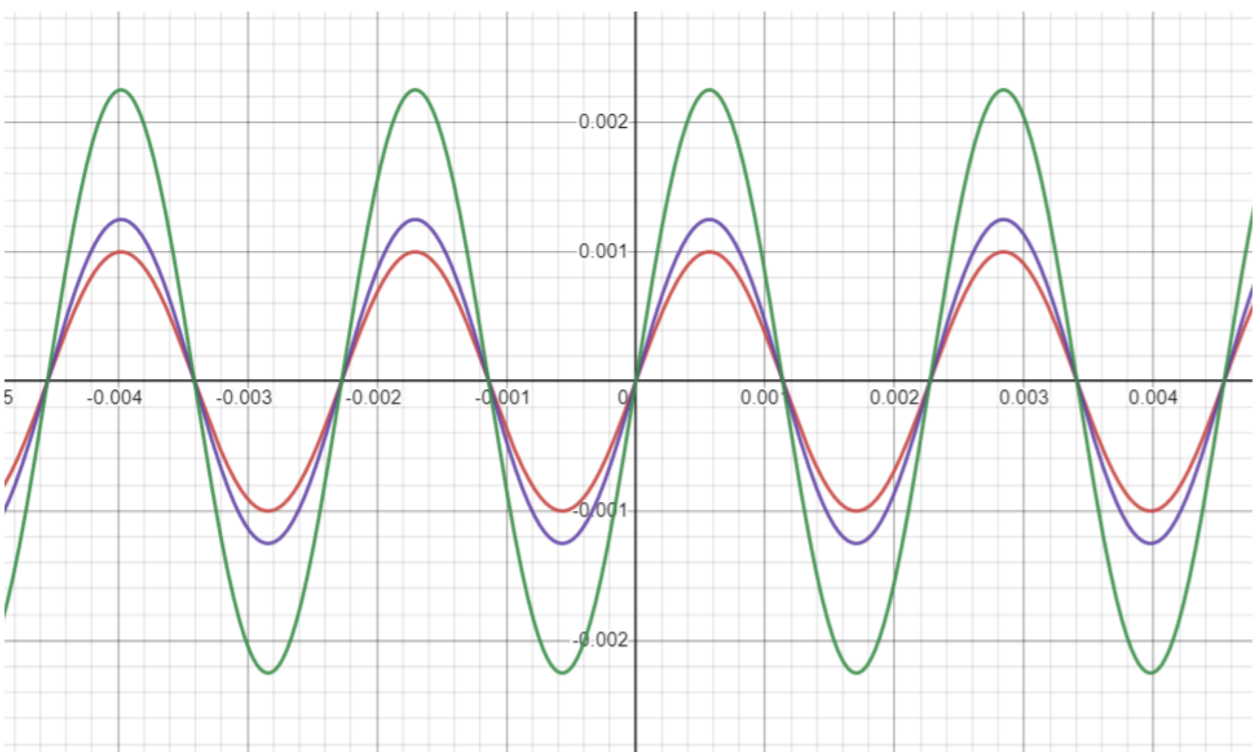
(The amplitude of the graph is set to 1/1000 so that the wave can be seen clearly when the graph was zoomed in.)

$$y = \frac{1}{1000} \sin(2\pi * 440\, x)$$

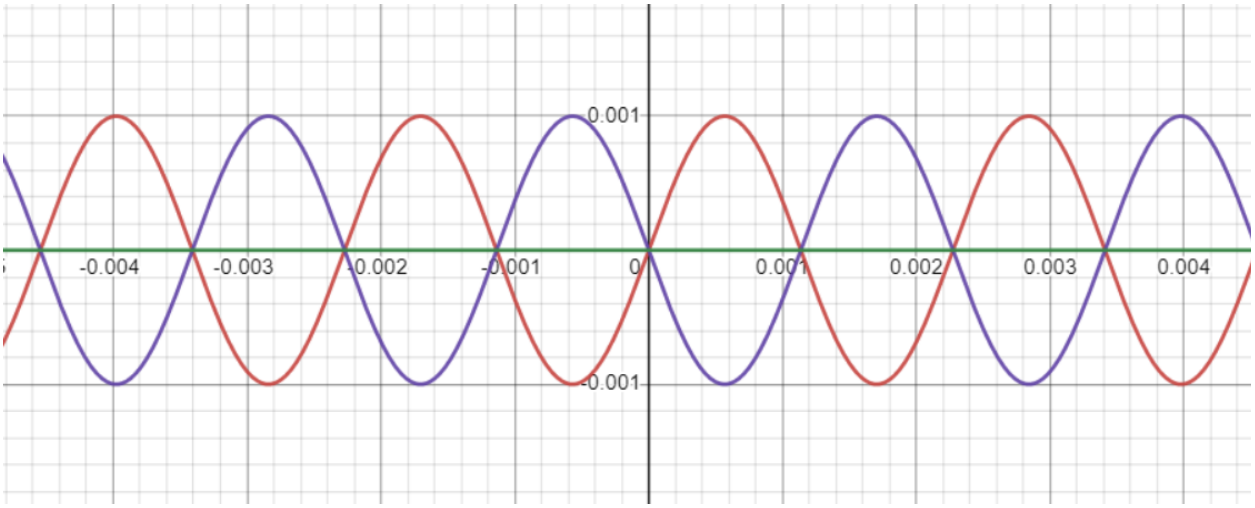


Superposing is the summation of the amplitudes of multiple sound waves. The sounds we hear are a summation of multiple frequencies of varying amplitudes. Instruments may play the same ‘note’ but have different timbre or overtones due to the summation of the other frequencies created. A flute and violin can play the note but have different qualities about the sound.

The concept of full constructive interference is when two waves of the same frequency, that are in phase to one another, superpose and their amplitudes sum to become a larger amplitude but maintain their common frequency.



Similarly, if two waves of the same frequency are exactly π radians out of phase, they will have full destructive interference. Destructive interference is used to our advantage in noise reduction systems such as in headphones or in the design of concert venues.



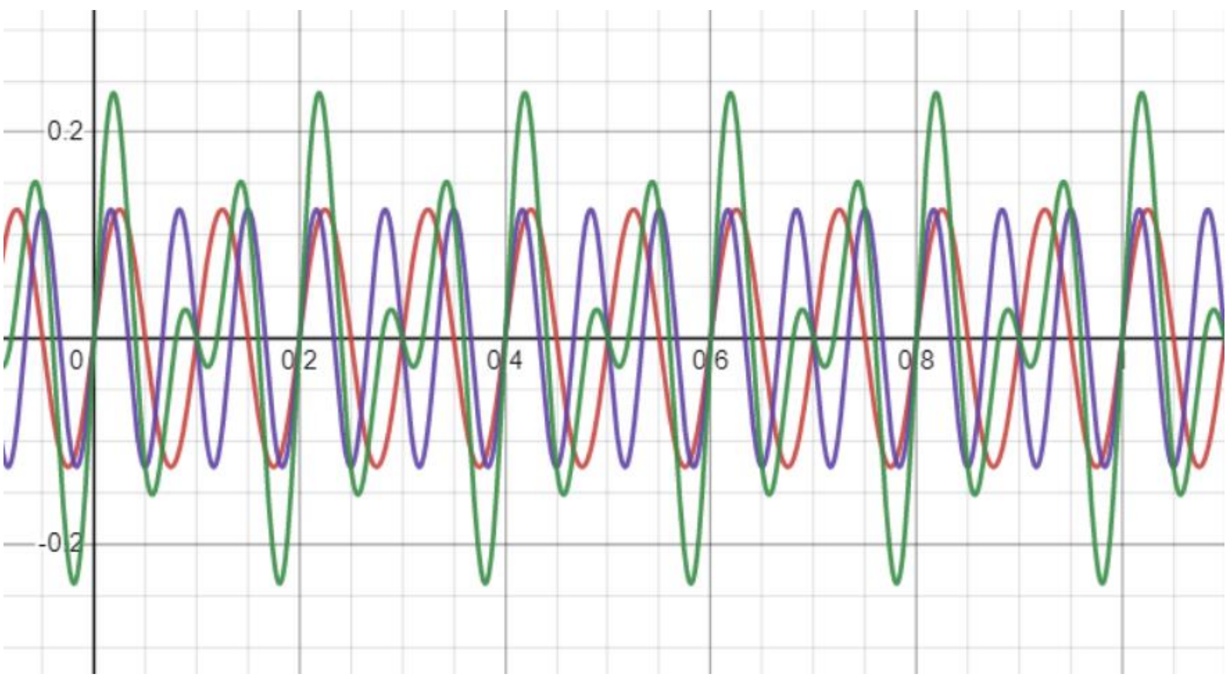
But what happens if you have two waves that have different frequencies?

Let's try two waves of 5Hz difference in frequency (with the same amplitudes).

Red = 10Hz

Purple = 15Hz

Green = superposition of the two waves



The green wave is the superposition of the two different sound waves. The superposing of the waves results in an oscillation of high and low amplitudes. As can be seen in the graphs, the waves with a larger difference in frequencies oscillate more times per second. The relationship between frequencies is such that the waves that differ by 5Hz oscillate 5 times per second.

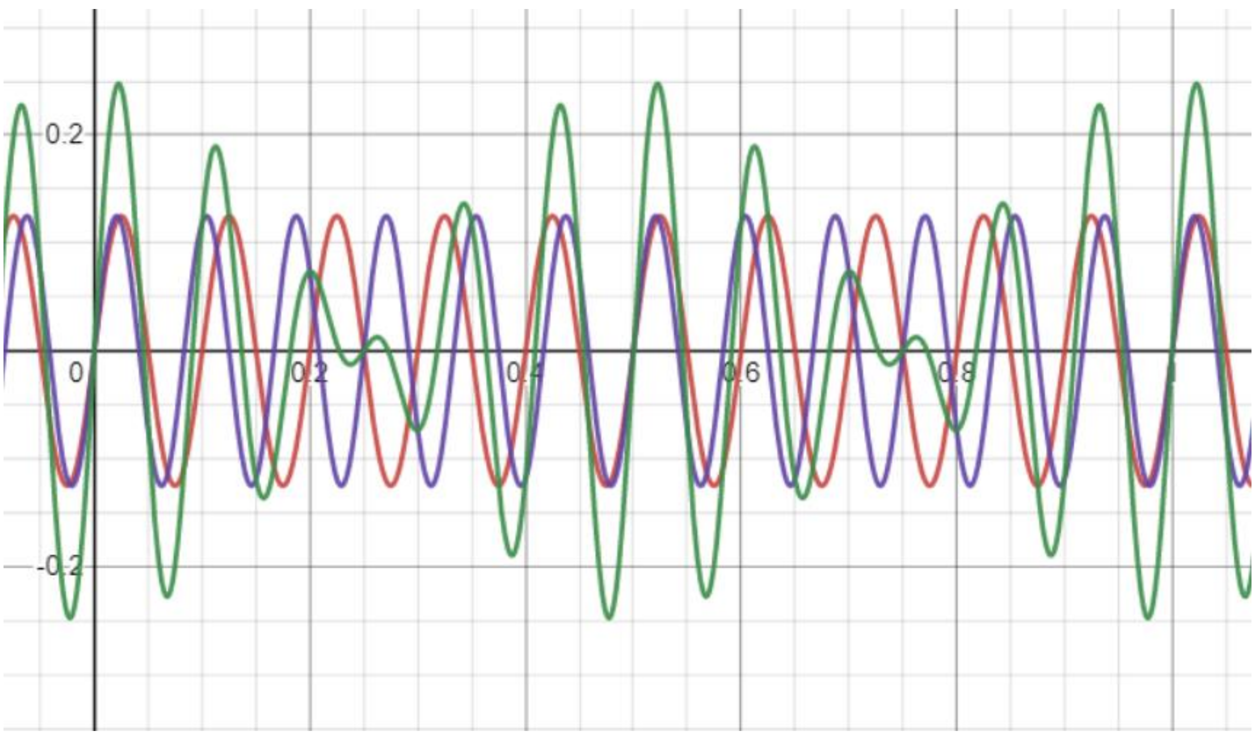
Compare that to two waves of 2Hz difference in frequency (with the same amplitudes).

Red = 10Hz

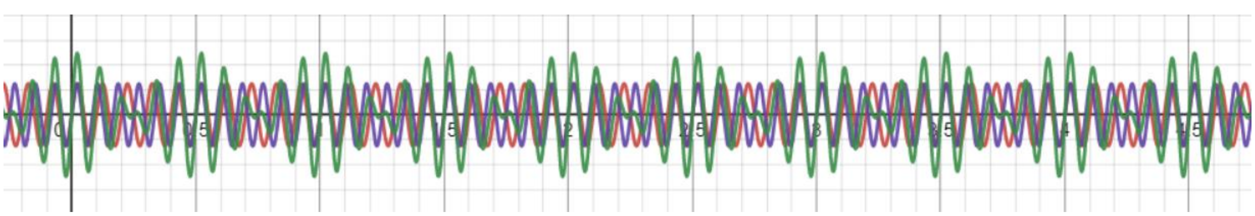
Purple = 12Hz

Green = superposition of the two waves

Now notice the frequency of oscillation from maximum to minimum amplitude is only 2 times per second.



When you zoom out, the oscillations are more evident.



Beat frequency

Beats refer to the recurring and regular variations in the amplitude of a sound resulting from the interaction between two sound waves of *nearly* identical frequencies. Beat frequency refers to the rate at which the volume is heard to be oscillating from high to low. For example, if 4 complete cycles of high to low oscillations are heard every second, then the beat frequency is 4Hz.

The beat frequency can be calculated by:

$$f_{beat} = |f_1 - f_2|$$

The formula can be derived from the equation for the superposition of the two original sine waves.

$$y_{1+2} = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$$

Here, the specific frequencies of the waves have been replaced by f_1 and f_2.

Using the following trigonometric identity, and the following equation:

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Our values for x and y will be:

$$x = 2\pi f_1 t \qquad\qquad y = 2\pi f_2 t$$

This equation can be produced:

$$y_{1+2} = 2\sin\left(2\pi t\frac{f_1 + f_2}{2}\right)\cos\left(2\pi t\frac{f_1 - f_2}{2}\right)$$

This equation reveals why beat frequency occurs. The sine term shows that the wave is partly a sine wave with the frequency of the average of f_1 and f_2. But the cosine term determines the repetition of the graph since it is the multiplier. The closer the frequencies are together, the closer the cosine term simplifies to 1. But if the frequencies differ, the cosine term will oscillate in value from maximum to minimum amplitude and will determine the overall frequency of repetition. It doesn't matter which is larger, f_1 or f_2, because the waves will overlap cyclically.

Therefore, the beat frequency is the absolute value of the difference in frequencies and only the cosine term defines the nature of the oscillations.

$$f_{beat} = |f_1 - f_2|$$

If the two instruments differ by a frequency of 2Hz such as 440Hz and 442Hz, then we will hear oscillations from highest to lowest volumes 2 times every second. When instruments are in tune with the note A, which has a frequency of 440 Hz, the beat frequency is zero and no fluctuations in volume are heard. The closer the instruments are to the desired frequency, the lower the beat frequency.

This phenomenon can be demonstrated by tapping two tuning forks of nearly identical frequencies and listening to the fluctuations of amplitude. Using this method, piano tuners can tune pianos. They pluck a piano string and then tap a tuning fork and if beats can be detected, then the two do not have identical frequencies. The piano tuner then adjusts the tension on the piano string until the fluctuations of amplitude can no longer be heard.

Here are some applications of beat frequency in addition to tuning musical instruments:

- Police RADAR: Doppler effect can shift the frequency of reflected microwaves. Beat frequency is then used to determine the relative speeds of vehicles.
- Interferometry: Beat frequency can be used in interferometry, a technique used to make precise measurements of distances and angles. Interferometers use the interference pattern created by the overlap of two or more waves to make these measurements. By detecting the beat frequency of the interference pattern, precise measurements can be made.
- Frequency modulation: Beat frequency can also be used in frequency modulation (FM) radio. In FM radio, the audio signal is used to modulate the frequency of the carrier wave. When two FM signals with slightly different frequencies are received, they produce a beat frequency that is equal to the difference in their frequencies. This beat frequency is then demodulated to recover the original audio signal.

The Fourier Transform

Where superposing adds waves together, the Fourier Transform breaks down waves into components of sinusoidal wave functions. Any wave (sound, electromagnetic fields etc.) can be rewritten as a summation of sine or cosine waves of various amplitudes and frequencies.

A good place to start is the Fourier Series which transforms a **periodic** wave into its components. Let's look at a square wave with a constant period T.

We can break down this square wave into sine and cosine waves of

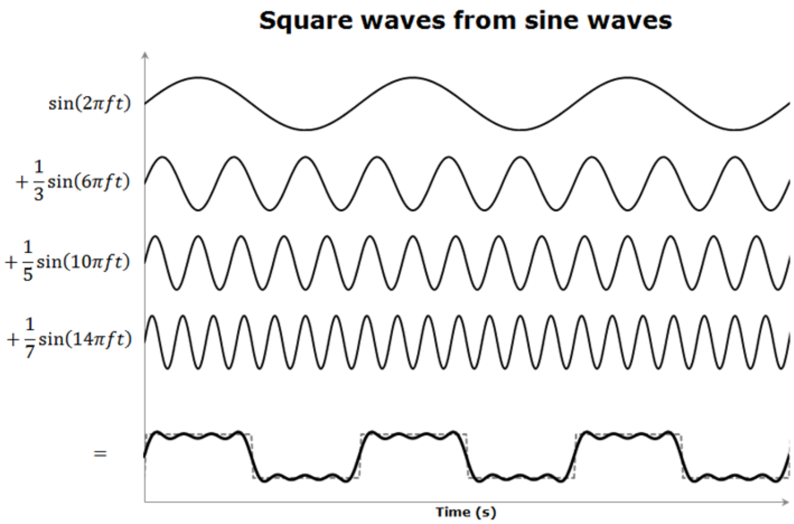


Figure 1. Generating a square wave from a sum of sine waves from: *THE PULSAR Engineering*. Available at: https://www.thepulsar.be/article/generating-sine-wave-from-square-waves/ .

The square wave can be broken down into a summation of sine and cosine waves of the equation:

$$f(t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin\big(2\pi f t(2n+1)\big)}{(2n+1)}$$

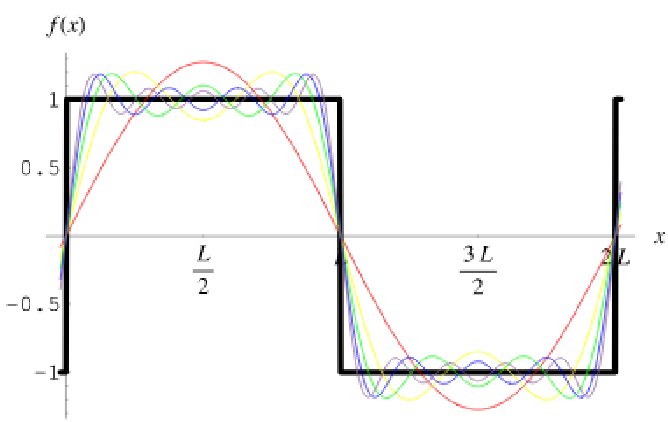


Figure 2. Fourier Series Square Wave from: *Fourier series--Square Wave from Wolfram MathWorld*. Available at: https://mathworld.wolfram.com/FourierSeriesSquareWave.html .

Once we have broken down the wave into its sinusoidal components, we can then plot the components showing the amplitude and frequency of each component. The Fourier Series of these set of equations looks like this:

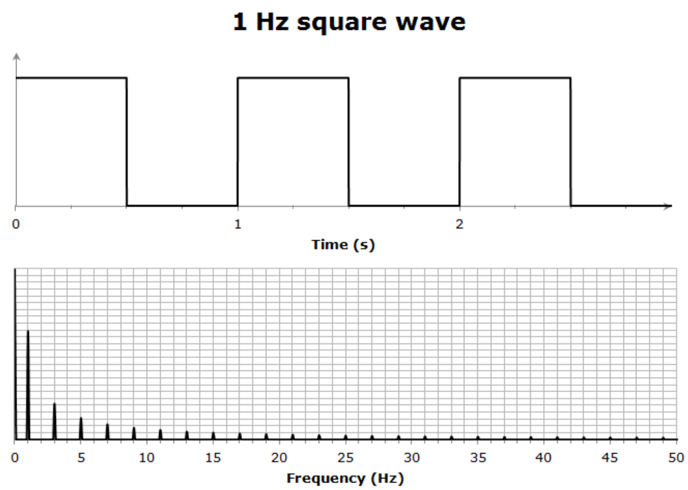


Figure 3. Fourier Transform frequency graph from: *THE PULSAR Engineering*. Available at: **https://www.thepulsar.be/article/generating-sine-wave-from-square-waves/ .**

Similarly, there is the Fourier Series of equations for the periodic triangular wave, sawtooth wave, semicircular wave which is just the component sine and cosine waves.

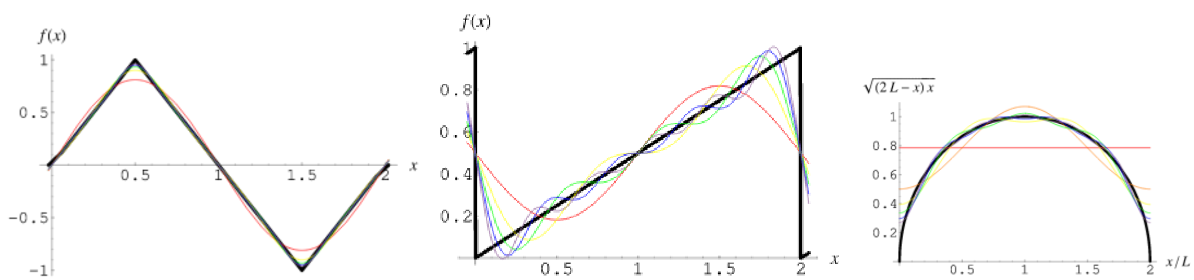


Figure 4. Fourier Series triangular wave, sawtooth wave and semicircular wave from: *Fourier series--Semicircular Wave from Wolfram MathWorld*. Available at: https://mathworld.wolfram.com/FourierSeriesSemicircle.html

A Fourier transform is used to analyze more complex waveforms than just these periodic ones and can determine the underlying frequencies that make up a signal. It takes a time-domain signal (time on the x-axis), which is a representation of how the signal changes over time, and converts it into a frequency-domain representation, which shows the intensity of each frequency component in the signal - sort of like a bar chart (frequency on the x-axis). In other words, it breaks down a signal into its constituent frequencies, allowing us to identify the specific frequencies that are present and the strength of each frequency component. The Fourier transform has many applications in various fields, including engineering, physics, and data analysis.

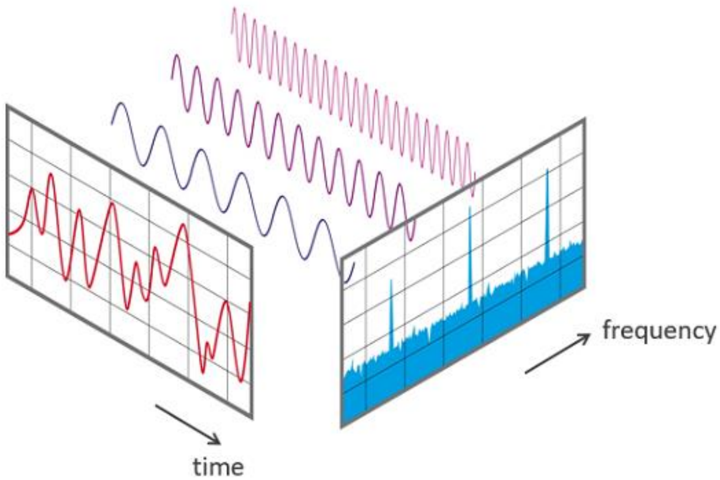


Figure 5. Fourier transform from: *FFT*. Available at: https://www.nti-audio.com/en/support/know-how/fast-fourier-transform-fft .

Some examples of Fourier Transforms:

- Time-frequency analysis of sounds produced by dolphins, or bats, can lead not only to identifying species but learning more about how these mammals communicate with each other and adapt to their environments.

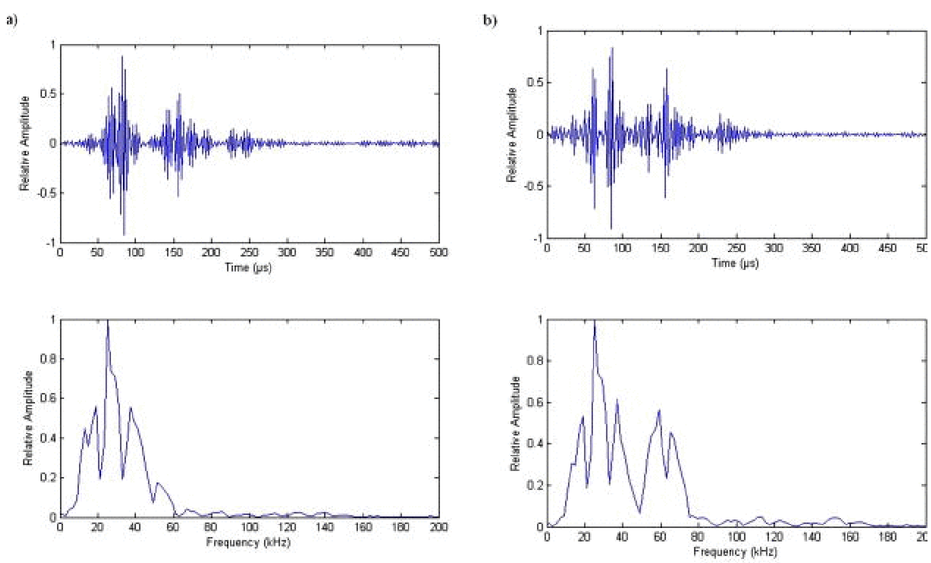


Figure 6. Time-frequency analysis of dolphin echolocation from: Muller, M.W. *et al.* (1970) *Time-frequency analysis and modeling of the backscatter of categorized dolphin echolocation clicks for target discrimination*, *Scitation*. Acoustical Society of AmericaASA. Available at: https://asa.scitation.org/doi/full/10.1121/1.2932060.

- Oxford scientists used Fourier transform analysis of spectroscopic data to analyse and identify bacteria in *Staphylococcus* Species Identification by Fourier Transform Infrared (FTIR) Spectroscopic Techniques: A Cross-Lab Study.
- Fourier transform analysis was used during the Cold War to distinguish nuclear testing from earthquake vibrations.

Fourier transformations of complex information signals take a lot of computing power and time. An algorithm to cut down the computing time is called the Fast Fourier Transform (FFT) which reduces the number of computations needed. The FFT is known to be one of the most useful mathematical algorithms of our time with its use in engineering and computing.

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