Chaos Theory: Determinism in Disguise

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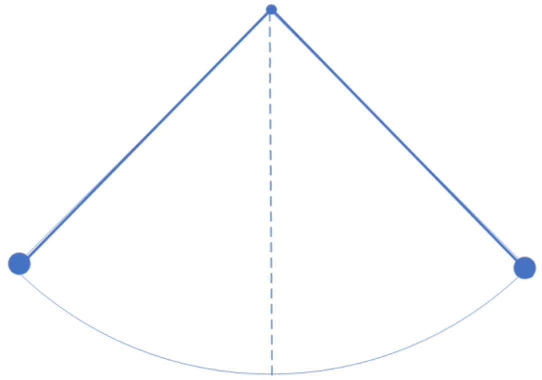
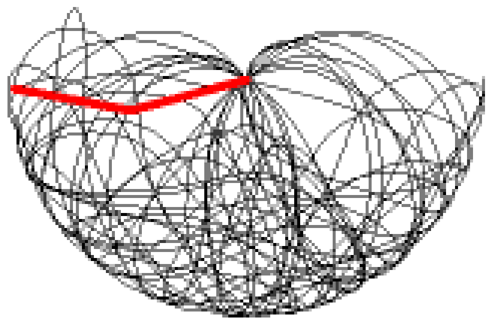
It is often flippantly remarked that 'the flap of a butterfly's wings in Brazil can trigger a tornado in the state of Texas.' This slightly cliche sentiment has been used widely to characterise what's known as the butterfly effect (for different reasons!), or as mathematicians recognize it: chaos theory. So, what is chaos theory? What problems does it solve? Why do mathematicians find it so intriguing? And more importantly, what do butterflies have to do with it?

Chaos was summarised by Edward Lorenz, a main character in the mathematics of chaos theory, as 'when the present determines the future, but the approximate present does not approximately determine the future.' Let's break that down. Firstly, it's about determinism. A chaotic system is one which is completely deterministic - the same input will *always* yield the same output. But – and here's the kicker - slightly differing initial conditions will cause results to diverge rapidly. The single pendulum (below left) and double pendulum (below right) can be used to illustrate the difference between a *stable* and a chaotic system. The single pendulum, when dropped, will follow a very predictable path, accelerating downwards due to gravity before swinging up to a maximum point on the other side. Most people would be capable of anticipating the path of a single pendulum and using common sense to reason that increasing the drop height would merely increase the maximum height reached proportionately on the other side. Now let's consider the less trivial double pendulum - a place where common sense doesn't belong. Not only is the pendulum's acrobatic movement comically unintuitive, but an infinitesimally small change in drop height or even temperature, humidity, you name it… will cause a complete change in the pendulum's path that's apparent after just a few seconds.





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Systems of this nature are characteristically difficult to predict as any slight discrepancy in measurement (which is inevitable as we cannot measure things to the required infinite resolution) will increase exponentially with time making it practically pointless to forecast results greater than a certain amount of time ahead. This amount of time is called the *predictability horizon*, and for the weather - one of the most recognised and studied chaotic systems - the predictability horizon is normally about 4 to 5 days. Systems that are more chaotic can have even shorter predictability horizons. The predictability horizon of a given chaotic system can be calculated with the help of the Lyapunov exponent, the number effectively dictating *how* chaotic a system is, or how quickly the distance between trajectories diverges. Or put more simply – the magnitude of an error made in prediction. Hold tight. More on this later.

A characteristic feature of chaotic systems is that the distance between trajectories, or the magnitude of an error, increases exponentially with time. Hence, we have:

$$d_t = d_0 e^{\lambda t}$$

where d_t is the distance between trajectories after t units of time, d_0 is the initial distance between trajectories, *t* denotes time and *λ* signifies the Lyapunov exponent. As you have likely observed, the size of the Lyapunov exponent determines the rate of exponential growth of the distance between trajectories and a greater Lyapunov exponent will lead to faster divergence of outcomes within the system. Note that since a defining feature of a chaotic system is that the distance between trajectories will increase exponentially, for a system to exhibit chaos its Lyapunov exponent must be positive, otherwise *e* will have a negative exponent and outcomes will converge. In the case that $\lambda = 0$, it can be seen that the trajectories remain at a fixed distance from each other since $d_t = d_0$ and hence, such a system is also not chaotic.

Now we understand what is meant by a system's Lyapunov exponent, we can understand how weather and road traffic forecasters along with traders predicting the stock market and data scientists predicting population statistics can decide the length of time after which their projections will no longer be accurate. The predictability horizon of a system (*P*), with maximum acceptable error (*a*), is given by:

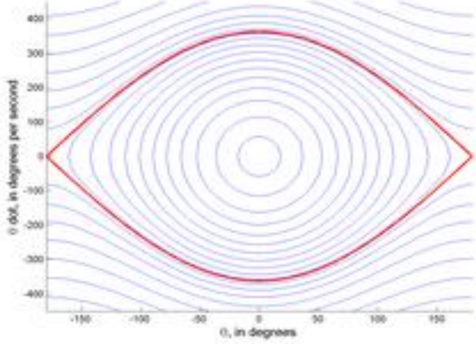
$$P = \frac{1}{\lambda} \ln \left(\frac{a}{d_0}\right)$$

Just to demonstrate how rapidly errors augment we can play around with the predictability horizons of a hypothetical system with Lyapunov exponent of 0.9 – this is approximately equal to the Lyapunov exponent of the chaotic system modelling the world's weather. Let's suppose we measure the initial conditions of our system with an error of 1×10^{-6} arbitrary units. Now consider a second measurement of the system, this time with an error of 1. Supposing we choose our maximum allowed error to be 1000, by the above formula we can see that the respective predictability horizons in these scenarios are $\frac{1}{0.9} \ln \left(\frac{1000}{1\times10^{-6}}\right) \approx 23$ and $\frac{1}{0.9} \ln \left(\frac{1000}{1}\right) \approx 7.7$.

So, the time taken for an error to increase by a factor of a billion is only three times greater than the time taken for an error to increase one thousand-fold? If that isn't chaos, then I don't know what is!

Now, it's about time I introduced you to Edward Lorenz, the so called *father of chaos theory*. Lorenz was an American mathematician and meteorologist, who spent time working as a weather forecaster. He realised that changes in weather, across just a few days, after having initial conditions measured to be approximately equal (you can see where this is going), bore little or no resemblance to each other. This crucial realisation formed the beginnings of what we now know as chaos theory, and more immediately, led to the development of the Lorenz system: a set of differential equations, that when plotted in a way where the system's changing state is represented visually, with the correct constant parameters, forms what has become the face of chaos theory and is what gives it the name 'the butterfly effect'.

In order to understand the Lorenz system on even a surface level (and to appreciate it on any level), we must first grapple with the idea of a *phase space* being used to represent a *dynamical system*. First things first, a dynamical system is all to do with rates of change, rather than coordinates. More specifically, a dynamical system is any system that can be written in the form $\frac{d\vec{x}}{dt} = f(\vec{x}, K_1 \dots K_p, t)$, where $\vec{x}$ represents the vector state of the system, *t* represents time and $K_1 \dots K_p$ represent constant parameters of the system. Really, this means that the way the state is progressing at a given point is just a function of the current state of the system, time elapsed and various other factors the system may rely on. Essentially, any system that exhibits change over time is dynamical, take for example, our single pendulum model from earlier. The state of the system - the angle θ between the equilibrium position and the pendulum, and the speed of the pendulum $\frac{d\theta}{dt}$, (note that the state of a dynamical system can comprise several unknowns; the Lorenz system has 3) changes with time and the rate of change of the state with respect to time can be computed with θ, $\frac{d\theta}{dt}$ and *t*. If we observe this system in a *phase space*, we see the following:



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Phase space is a cartesian area that allows us to visualise how a dynamical system evolves by demonstrating the direction each possible set of conditions will travel in. For dynamical systems with states consisting of multiple variables (that is to say most of them!), the direction of the trajectory at a given point is the direction of the vector sum of all rates of change of the variables, and arrows are often implemented to outline the direction of every trajectory. When plotting dynamical systems in phase space, a variety of different patterns known as *attractors* can appear. An attractor usually (but not always) is a point that attracts nearby trajectories; all trajectories within that attractor's *basin of attraction* will collapse into the attractor - not unlike how all rainwater and streams in a river basin eventually feed into one central river, just in case a fluvial analogy helps. The simplest type of attractor is a fixed point attractor, one of which is apparent in the centre of the above phase space of a pendulum, representing the equilibrium position of the pendulum. Fixed point attractors tend to materialise in non-chaotic systems, while the type of attractors we're interested in, *strange* attractors, are how chaos likes to manifest in phase space, and are what inspired Edward Lorenz to spend a large proportion of his life studying deterministic chaos.

The Lorenz system is the set of differential equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

The equations exhibit very chaotic behaviour for some values of σ, ρ and β, specifically, values close to the values Lorenz used: $\sigma = 10,\ \rho = 28$, and $\beta = \frac{8}{3}$. The system models fluid dynamics in such a way that x is proportional to the rate of convection and y and z are proportional to the horizontal and vertical temperature variation respectively. Variations on the model have also been used to explain the chaotic behaviour of weather, chemical reactions, osmosis and various other dynamical systems. When this is represented in a 3 dimensional phase space, we see the famous Lorenz attractor, and this is why chaos theory is often called the 'butterfly effect'; it has nothing to do with our little friend in Brazil!

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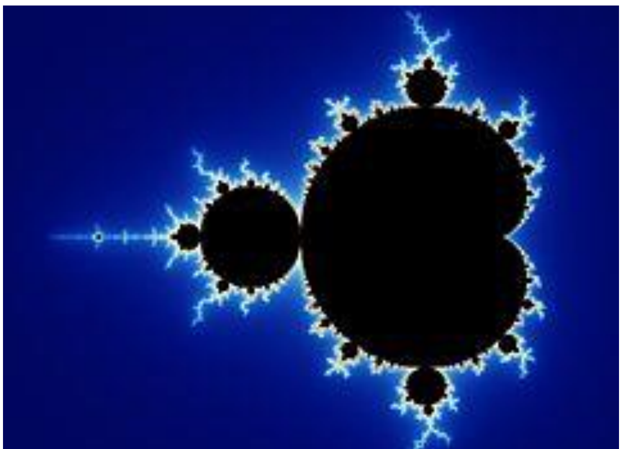
Beautiful isn't it? But don't stop for too long; not only are the Lorenz attractor's hypnotic spirals aesthetically pleasing, but many aspects of this phase space plot are also food for thought for the inquisitive pure mathematician. The Lorenz attractor is an example of a strange attractor - a type of pattern that may appear in a phase space plot of a dynamical system. Unlike the fixed point attractor seen in the plot of the single pendulum - a singularity towards which all points in the basin of attraction move through an area called the 'transience'- the strange attractor has an infinitely complex, *fractal* structure. This means any infinitesimally small movement across the attractor will meet another unique trajectory with vastly different plans to the adjacent trajectory. This of course translates to the high sensitivity to initial conditions we clearly observe in real life chaotic systems. Similar chaotic behaviour is exhibited by the infamous gingerbread man-shaped fractal, the Mandelbrot Set (you have to want to see a gingerbread man!), shown below left. The Mandelbrot is the set of complex numbers - numbers formed from the addition of imaginary numbers (multiples of $\sqrt{-1}$) and real numbers - that don't diverge to infinity when iterated through a certain function. The black area shows numbers that don't diverge - the Mandelbrot Set - while the blue area represents numbers that do. Most of the surface area of the fractal exhibits completely stable behaviour while the boundary of the set is completely chaotic and infinitely complex, meaning any infinitesimally small movement in any direction in the chaotic boundary will cause a number to diverge instead of converging, much like the Lorenz attractor. The fractal is also famous for being self-similar no matter how far one zooms in (below right) - another example of beauty and order in chaos.



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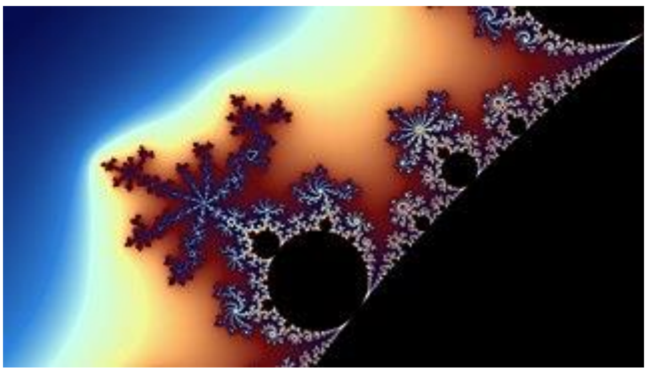

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So, there you have it, a brief(ish) explanation of chaos theory - one of the most beautifully confusing ideas in all of mathematics, but also one of the most important. Chaos underpins everything that happens to us, from being caught in a rainstorm or losing money as share prices fall, to the very evolution of our solar system and universe. Chaos is all around us. But it is most definitely not randomness. In fact, it was Pierre Simon Laplace that once said 'give me the positions and velocities of all the particles in the universe, and I will predict the future'. While there are some slight practical issues with that statement, the message should hopefully by now be clear; there is a very real dichotomy between predictability and determinism. Chaotic systems *appear* completely random on the outside while they are in fact entirely deterministic. But this doesn't mean we can predict them accurately.

So what *practically* does the mathematics of chaos theory give us then if not accurate predictability? It gives us the tools to model and understand chaos in ways that let us explain the behaviour of the world around us and ascertain to what degree we *can* predict *reasonably enough* the different systems at play. The study of chaos theory also gives rise to some weighty questions like 'is anything truly random?' and if not, 'is the entire universe pre-decided?'