<u>Circle Inversion and its Uses</u>

<u>A Mathematics Essay by Adi Srivastava</u>

Circle inversion is also known as the 'Dark Art of Mathematics', but why is this?

Before we discuss this name, we must first understand what circle inversion is and what its uses are.

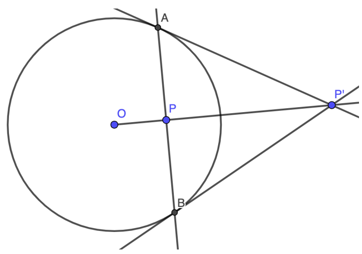
<u>What actually is circle inversion?</u>

Circle inversion is a transformation which requires two parameters: a centre of inversion (represented by the letter 'O') and a radius of inversion, which has length 'r'. Putting these together, we form a circle of inversion with centre O and radius r, hence this transformation is called 'Circle Inversion'.

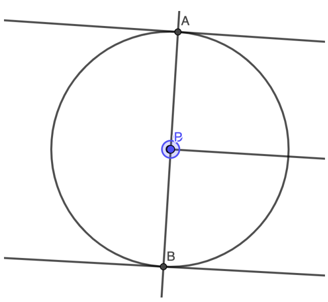
<u>How is it done?</u>

First let us focus on certain points. There are four different possible locations the point could be with respect to the circle of inversion.

1. The point, P, is inside the circle, but not at the centre. For this case, we start by drawing the ray from O through P. After this, at P we draw a perpendicular line to this ray, and draw tangents to the circle where this line intersects it (points A and B in the diagram). Due to rules of tangents, radii, and congruences, these tangents will meet the ray at the same point. This is our inverted point, P'.

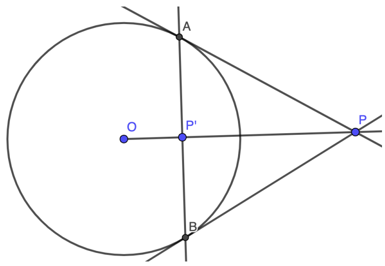


2. A special case of 1: the point, P, is at the centre, O. Using the same method as above, we first try to draw a ray from O through Q - but this is any ray from O. So as a result our perpendicular will be any diameter, and our tangents any two *parallel* tangents of the circle. However, we want to find where these converge. Parallel lines never meet, but we can say that they meet infinitely far away (in this essay I will shorten this to *'the centre goes to infinity'*). Putting this together, we get that Q goes to any point at infinity.

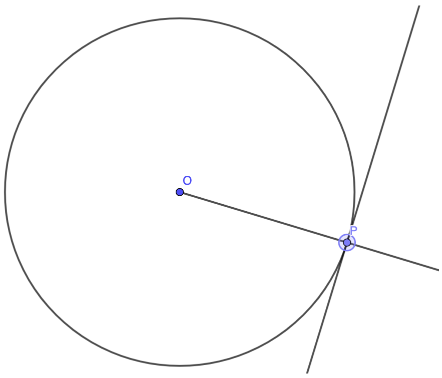


3. The opposite of 1: the point, P, is outside the circle. For this, we start by drawing the ray from O through P, but for the rest, we reverse the order of

actions. We must *first* draw the tangents from P to C and D (points on the circle), *then* we connect them (forming a perpendicular), and *finally* we label the intersection between the ray and perpendicular as the inverse point P’. This results in the concept that (P’)’ = P.

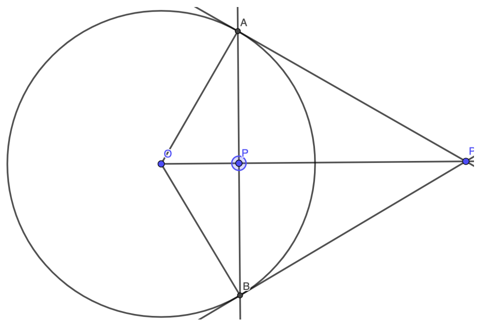


4. A special case of both 1 and 3: the point, P, is *on* the circle. For this, we can try both methods. Using 1, we draw the ray, then draw a perpendicular - and we get that our points from where we draw tangents, E and F, *are the same point,* P. As a result, the tangents are the same, so they will intersect the perpendicular at P. So P is the same as P’. Using 3, we draw the ray and tangents, which meet the circle at E and F, both of which actually are P. Thus the line which connects E and F intersects the ray at P. So S is the same as S’.



<u>A simplification</u>

From here, we must start to integrate formulae. Let us first find a formula to find the inverse point given the original point P. Take the diagram below (here we are assuming that P is inside the circle, as 2 and 4 fall under this category, 1, and for 3, I will explain later). Now following the steps from above, we get the diagram to the righ (note that OA and OB are also drawn in). We now see that 90 - PÂO = PÂP’ = 90 - AP’P. So PÂO = AP’P. So by AA, OAP is similar to OP’A. So OP/OA = OA/OP’. This is the same as OP/r = r/OP’, and multiplying through by OP’ x r, we get:



OP.OP’ = r^2

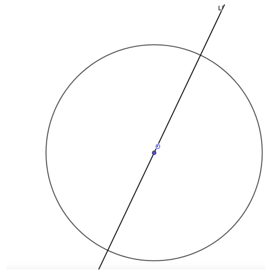
This is the most important formula in circle inversion, as not only does it help find new formulae, it also *defines* how it works, explaining placements of points in space.

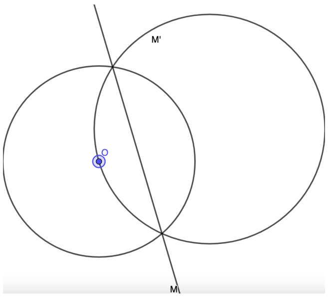
Also note that if P were to be *outside* the circle, the formula would be OP’.OP = r^2. As all products are commutative, this results in the same formula as before. So now we have taken into account all possible placements of point P.

<u>Moving to straight lines …</u>

When inverting straight lines, we have two possible cases.

1. The line, L, goes through the centre. This goes to the same line L, as each point on L goes to another point on it. Also, L goes to infinity, and goes through O, so L’ goes through those as well. As a result, L = L’.



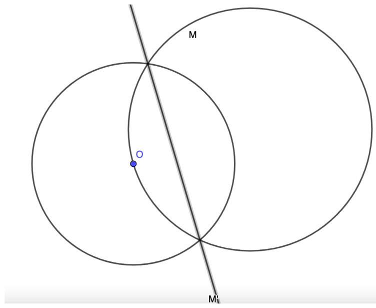


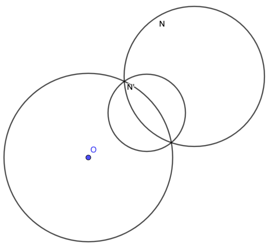
2. The line, M, does *not* go through O. This will end up as a circle of inverse points, but it also must be noted that all lines, including M, go to infinity, so the circle M’ goes through the centre O. Importantly, the diameter of M’ through O is *perpendicular* to M.

<u>… and circles</u>

This time, we actually have three cases.

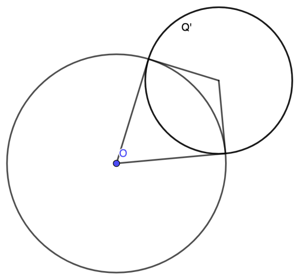
1. The circle, M, goes through O. As we have seen earlier, an inverse of an inverse is the original, so following our rules from above, if a straight line not through the centre goes to a circle through the centre, then a circle through the centre goes to a straight line not through the centre, perpendicular to the diameter through O.



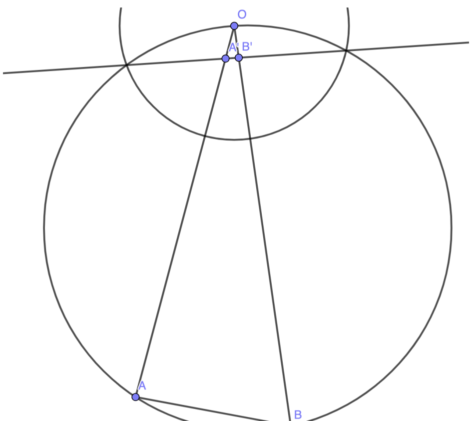


2. The circle, N, does *not* go through the centre, but is *not* orthogonal to the circle of inversion. This goes to another circle, not through the centre (as N doesn’t go to infinity).

3. A special case of 2: the circle, Q, does not go through the centre, and *is* orthogonal to the circle of inversion. For this, following 2, we get to a circle, but here, we get to *the same* circle. Here, Q = Q’.



<u>Another important formula</u>



This takes this diagram (to the left) and finds an inverse length from original lengths (remember that A’ and B’ lie on the rays from O to A and B respectively and that the circle through O inverts

to a straight line not through O). r is the radius of inversion.
Now:

$$OA.OA' = r^2$$
$$OB.OB' = r^2$$
$$\text{So } OA.OA' = OB.OB', \text{ and } OA/OB = OB'/OA'.$$
$$\text{And } A\hat{O}B = B'\hat{O}A'.$$

So by an angle and a common ratio:

$$AOB \sim B'OA'.$$
$$A'B'/AB = OB'/OA.$$
$$A'B' = AB.OB'/OA.$$

But this contains a new value on the right hand side; we only want original ones. But using our other formula,

$$OB' = r^2/OB.$$
$$\text{So } A'B' = AB.r^2/OA.OB.$$

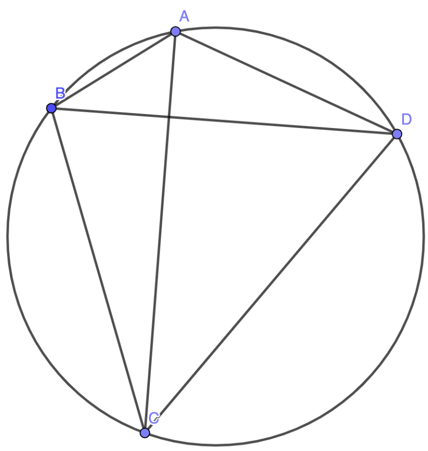
This is an extremely important equation in the realm of circle inversion, as will be demonstrated later.

And why the dark art of maths?

Circle inversion is so powerful in proof that it is called this name. It can be used to simplify difficult equations into easier ones, as will be demonstrated. Examples of complex things which can be proved are Feuerbach's Theorem, and the one we will focus on and turn into a line rule which most 6-year olds know, Ptolemy's Theorem.

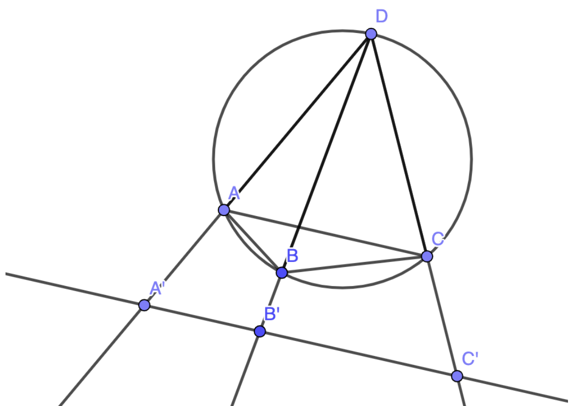
Ptolemy's Theorem: What is it?

To the right, we have a cyclic quadrilateral, ABCD. Ptolemy's Theorem states that AB.CD + AD.BC = AC.BD. But this works for any cyclic quadrilateral. But how can this be proven?



A proof

Of course, as is the focus of this essay, we will prove it by inversion, specifically the second formula we have proven. Imagine D being the centre of inversion, and a length r as the radius (this doesn't matter, I will explain later). We get a diagram similar to that in 'Another Important Formula', but with three points, A, B and C (to the left). Using our formula, we get that:



$$A'B' = AB.r^2/OA.OB.$$
$$B'C' = BC.r^2/OB.OC.$$
$$A'C' = AC.r^2/OA.OC.$$

But A’B’C’ is a straight line, so A’B’ + B’C’ = A’C’. Substituting our above formulae:

$$AB.r^2/OA.OB + BC.r^2/OB.OC = AC.r^2/OA.OC.$$

Now here we divide through by r^2, leaving no r terms, so r doesn’t matter.

$$AB/OA.OB + BC/OB.OC = AC/OA.OC.$$

And finally multiplying through by OA.OB.OC:

$$AB.CO + AO.BC = AC.BO.$$

But O is D. So:

$$AB.CD + AD.BC = AC.BD.$$

This looks quite familiar; we have proven Ptolemy’s Theorem.

<u>A Conclusion</u>

This proof clearly demonstrates the extreme power of circle inversion, but it also has its disadvantages. The main one is difficulty: it is extremely difficult to comprehend at first, as it is a transformation which flips space itself, as compared to others which just move an object *in* space. All in all, I believe that circle inversion, once understood, is an essential in any mathematician’s toolkit, due to its power and simplification of difficult problems.