Randomness in a Deterministic World:
The Double Pendulum

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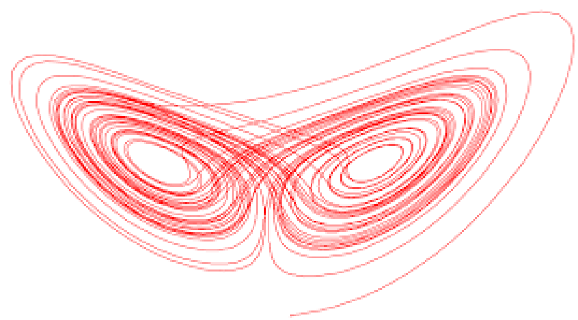
We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

— Pierre-Simon Laplace, A Philosophical Essay on Probabilities [1]

Left: Pierre-Simon Laplace, Right: The Lorenz Attractor

The concept of chaos has fascinated scientists from a variety of disciplines, as it is apparent in so much of the real world. Despite its mathematical beauty, much of its depth lies in the way it interpolates between determinism and randomness. How determinism appears to be lost is both important in many applied settings such as physics, engineering and ecology, as well as being mathematically complex. One classic example of a chaotic system is the double pendulum, which exhibits irregular motion that appears to be unpredictable. Despite this lack of predictability, the motion of the double pendulum is actually governed by clear deterministic laws. Here we will explore the double pendulum as a case study in how chaos can appear in simple deterministic systems, shedding light on the sort of mathematics involved.

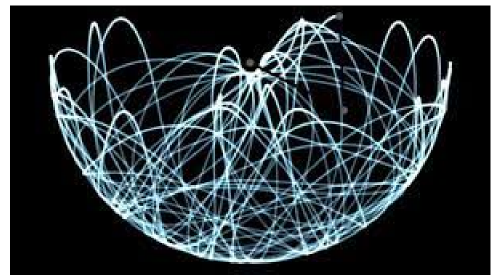
A chaotic system is one subsection of dynamical systems, whose configuration is described by a phase space (in that every possible state can be shown as a point on this phase space) that evolves with time, which can be either continuous or discrete. There are two broader types of dynamical system, stochastic and deterministic. We will focus on deterministic systems initially, where the future is uniquely determined by the past, so there are clear rules or formulae to calculate this.

One would suppose that an approximate value would lead to an approximate solution, but due to the nature of chaotic systems, from an approximate value there can be no accurate solution. From an approximate value, two different systems identical within the resolution of that approximation can and will behave entirely differently in the long run. This is because a minute difference in the past will result in a hugely different future, also known as the Butterfly Effect, as small perturbations in the initial conditions completely change the outcome. This sensitivity to initial conditions means that these initial conditions can never be measured accurately enough. In order for these models to be applied to real life, these starting values must be to an infinite precision, as being off by just a tiny percentage of the accurate value will cause the paths to diverge uncontrollably, so precise prediction is incredibly difficult, especially after longer periods of time.

To look deeper into chaos and our ideas of randomness, we can explore the double pendulum as an example of a chaotic (deterministic) system.

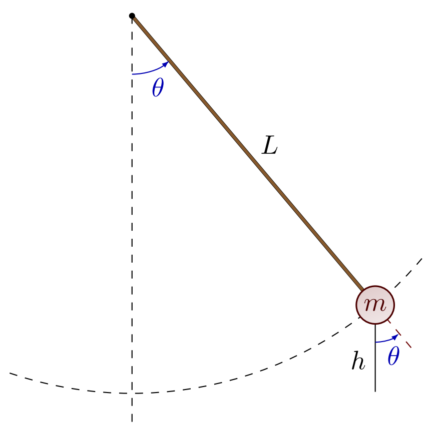
A single pendulum is a mass suspended from a pivot, so that it can freely move in any direction. For modelling purposes, we assume that the pivot is frictionless and the string connecting the mass and the pivot is light (ie has no mass) and inextensible. A double pendulum is just a single pendulum with another pendulum added onto the mass of the first pendulum. Now, since with a single pendulum there is one degree of freedom, with a double pendulum, as we now have two pivots (one at the top that the first pendulum is attached to and one connecting the two pendulums), there are two degrees of freedom, and this makes the system much more complex, and below we can see an image of the path of the second mass on a double pendulum.

However, not all systems with two degrees of freedom are chaotic, as in order to have chaos, the equations defining the system must be non-linear. This means that the variables (or, in the case of differential equations, the unknown functions) cannot be defined using equations without a polynomial term with a power greater than one. Sometimes non-linear systems can be estimated by linear equations (linearisation), but this does remove a lot of the accuracy.



The path of the mass of a double pendulum

Before discussing the differential equations of the double pendulum, let's first look at the single pendulum.



To do this, we must look at the relationship between τ, which represents the net torque, which is the measure of a force that causes an object to rotate about a pivot or an axis, I, which represents the rotational inertia, a property to do with the mass distribution of the system and α, which represents the angular acceleration. From these definitions, it is clear that this is a rotational analogue of Newton's second law $F = ma$.

$$\tau = I\alpha \tag{1}$$

By using this for the single pendulum, we can have an equation just defined by θ, as both x and y can be determined from this, so we only have to deal with one unknown variable. Since the pendulum string length is constant, we can use this coordinate system.

$$-mg\sin\theta L = mL^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{2}$$

Here we have assumed the pendulum is under the gravitational pull of the earth, with an acceleration due to gravity of g. This is a second order differential equation, due to the second derivative of θ with respect to time representing the angular acceleration.

We can linearise this using the small angle approximation (derived from the Taylor expansion), where $\sin\theta \approx \theta$, when θ is in radians and is small. There is no clear limit to what 'small' means, but as θ increases, this approximation becomes more and more inaccurate. By doing this we would get

$$-mg\theta L = mL^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{3}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\theta = 0 \tag{4}$$

We can solve this by letting θ be in the form e^{rt}. By replacing any forms of θ with their respective forms of this equations, we get

$$e^{rt}(r^2 + \frac{g}{L}) = 0 \tag{5}$$

And as e^{rt} cannot be 0, we can divide (6) by e^{rt}

$$r^2 + \frac{g}{L} = 0 \tag{6}$$

$$r = \pm\sqrt{\frac{g}{L}}\, i \tag{7}$$

So our two solutions for θ, using our two values for r, can be expanded using Euler's Formula reproduced here below

$$e^{it} = \cos t + i\sin t \tag{8}$$

We can use this in order to separate the real and complex components of our solutions,

$$e^{\pm\frac{\sqrt{g}}{\sqrt{L}}it} = \cos(\sqrt{\frac{g}{L}}t) \pm i\sin(\sqrt{\frac{g}{L}}t), \tag{9}$$

and using the principle of superposition, which states by adding two solutions (to a linear, homogeneous differential equation), you can find another root. Therefore we can add combine the two solutions in (9) to get another root, so removing the complex component of these solutions and we can then use ω to replace $\sqrt{\frac{g}{L}}$, with ω representing the natural frequency of this pendulum, and we must also include the initial angular displacement or θ_0, we therefore have

$$\theta(t) = \theta_0 cos(\omega t) \tag{10}$$

We can also look at the simple pendulum with Lagrangian mechanics, but first let's define what Lagrangian mechanics is and what it is used for.

Lagrangian mechanics is an alternative to Newtonian mechanics (or classical mechanics), which is based on Newton's laws of motion and we have used above. Lagrangian mechanics was introduced by Joseph-Louis Lagrange in 1788, and completely changed techniques surrounding mechanics. Lagrangian mechanics, instead of dealing with vectors and individual forces, instead focusses on energy within systems, by considering the kinetic and potential energy, so using only scalar quantities. Neither one is clearly better with the example of the single pendulum, but when getting to the double pendulum, since the individual forces are complex, using the Lagrangian is evidently the best approach.

The Lagrangian (L) is defined as below:

$$L \equiv T - V \tag{11}$$

Here T is the kinetic energy and V is the potential energy, so L is the difference between the kinetic and potential energy.

The key equation to do with the Lagrangian is below, the Euler-Lagrange equation. This stems from the principle of least action[1], which states that the trajectories are stationary points of the system's action function. Now the action (S) has the units joule-second and is represented by the definite integral with respect to time between two times (commonly t_1 and t_2) of the Lagrangian. The name of 'principle of least action' is a misnomer, as we are not looking just for minima, but also for any saddle points, but typically we do find where action is minimised.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{12}$$

$$S = \int_{t_1}^{t_2} L \,\mathrm{d}t \tag{13}$$

Now to work out the kinetic energy (KE), we cannot use the typical formula of $\frac{1}{2}mv^2$, since this would leave us with x and y components of v, and not in terms of θ like we need. To get KE in terms of θ, we must find another way of expressing KE. This comes from the rotational inertia (as seen above by using I) and the angular velocity (which, confusingly, has the same symbol, ω, as the natural frequency of a pendulum, but these are different quantities). $I = mL^2$ and $\omega = v/L$, so

$$\frac{1}{2}I\omega^2 \tag{14}$$

Therefore we can write KE as $\frac{1}{2}I\omega^2$, and as ω is the same as the first derivative of θ with respect to time (as it is the angular velocity, which is the rate of change of angular displacement, or θ), we can write KE as

$$T = \frac{1}{2}mL^2\dot{\theta} \tag{15}$$

and the dot above the θ indicates a derivative with respect to time.

To work out the potential energy, we must use the formula for gravitational potential energy (GPE) which is GPE = mgh, where m is the mass, g is the acceleration due to gravity and h is the height (and in this scenario we will define '0' to be at the resting position of the pendulum, where the mass is directly below the pivot). We then must work out h (labelled on the diagram of the single pendulum above) in terms of θ. This would be the y-coordinate, but we must find this in terms of θ, by taking away the vertical side of the triangle formed by the pivot, the mass and the point on that centre line in line with the mass, which would be $L\cos\theta$ and taking that away from the total length of L. The GPE, therefore, is

$$V = mgL(1 - \cos\theta) \tag{16}$$

The Lagrangian works out to be (15) - (16)

$$L = \frac{1}{2}mL^2\dot{\theta} - mgL(1 - \cos\theta) \tag{17}$$

$$= \frac{1}{2}mL^2\dot{\theta} - mgL + mgL\cos\theta \tag{18}$$

and if we calculate our example using the Euler-Lagrange equation, careful to make sure that we are taking the partial derivative where the ∂ symbol is shown, which means we differentiate with respect to only one variable and ignore any other variables, even if they may be implicitly defined with the same input variable, we get:

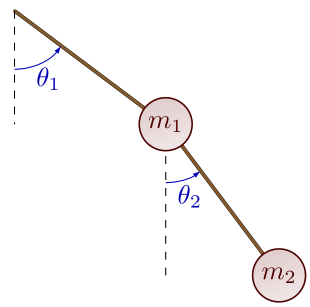
$$\frac{\mathrm{d}}{\mathrm{d}t}(mL^2\dot{\theta}) - (-mgL\sin\theta) = 0$$

$$mL^2\ddot{\theta} + mgL\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{19}$$

[1] For a detailed explanation of this principle and a slightly less detailed but still useful derivation of the Euler-Lagrange equation, see this video: [2]

And this is the same result we worked out with the Newtonian method as above, just with slightly different notation of the dots above the functions representing the time derivatives, and by repeating the technique of linearisation and solving the second order differential equation, we can again get the function θ.



Now onto the double pendulum. This is more complicated because we now are working with two functions θ_1 and θ_2, both labelled on the diagram above. In order to calculate the Lagrangian for this system, we can use the coordinates x_1, y_1, x_2 and y_2 for calculating T and V, and then replace these expressions in terms of θ_1 and θ_2.

To start with, the potential energy for the system is found by adding the gravitational potential energy of each of the masses, which are respectively m_1gh_1 and m_2gh_2. Here h_1 and h_2 are the y-coordinates of the masses. If we define the pivot to be at (0,0) again, these y-coordinates will be negative, and as g is negative (with a value of 9.8$ms{-1}$), V will be positive.

$$V = m_1gy_1 + m_2gy_2 \tag{20}$$

The kinetic energy of the system can be found by using $\frac{1}{2}mv^2$ and v here is equal to $\dot{x} + \dot{y}$, where $\dot{x}$ and $\dot{y}$ are the components of velocity in the x and y direction respectively. Again we can simply add the kinetic energies of the two masses. We cannot use our equation for KE that we did with the single pendulum, as the kinetic energy of the second mass is not just defined by θ_2, but also θ_1

$$T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) \tag{21}$$

Now we must calculate the coordinates of the masses in terms of θ_1 and θ_2. We can call the length of the first string R_1 and the length of the second R_2. For the coordinates of m_1, we can use trigonometric functions.

$$x_1 = R_1 \sin\theta_1 \tag{22}$$

$$y_1 = -R_1 \sin\theta_1 \tag{23}$$

Here y_1 has a minus sign in front as for angles of θ_1 between 0 and $\frac{\pi}{4}$ radians, where $\sin\theta_1$ is positive, y_1 is negative as it is below the pivot.

For m_2 we can do the same thing of $R_2 \sin\theta_2$ and $-R_2 \cos\theta_2$, but this gives us the coordinates relative to m_1, not relative to the stationary pivot, so to fix this, we must add our x_1 and y_1 values to these.

$$x_2 = R_2 \sin\theta_2 + R_1 \sin\theta_1 \tag{24}$$

$$y_2 = -R_2 \cos\theta_2 - R_1 \cos\theta_1 \tag{25}$$

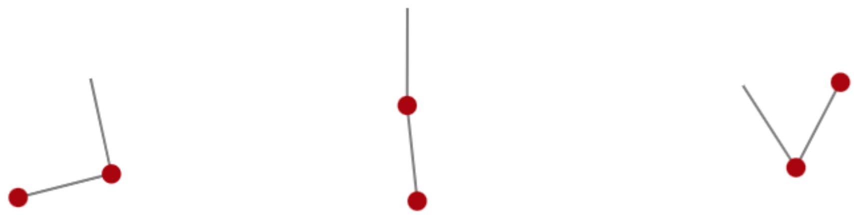
By repeating the process[2] as with the single pendulum, of calculating the Lagrangian and using the Euler-Lagrange equation, we get two second order differential equations. Here we get two equations instead of just one as above with the single pendulum, as we can use the Euler-Lagrange equation with both θ_1 and θ_2.

$$(m_1 + m_2)R_1\ddot{\theta_1} + m_2R_2\ddot{\theta_2}\cos(\theta_1 - \theta_2) - m_2R_2\dot{\theta_2}^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)g\sin\theta_1 = 0 \tag{26}$$

$$m_2R_2\ddot{\theta_2} + m_2R_1\ddot{\theta_1}\cos(\theta_1 - \theta_2) - m_2R_1\dot{\theta_1}\sin(\theta_1 - \theta_2) + m_2g\sin\theta_2 = 0 \tag{27}$$

[2] For a step-by-step walkthrough, see this video: [3]

Now, these are solvable, but since I do not want to double the length of this essay, I will use a computer to model these. By using Mathematica, we merely have to input these differential equations and set up the pendulum, and then we see an animated model of our pendulum, for whichever starting angles and velocity we choose[3].

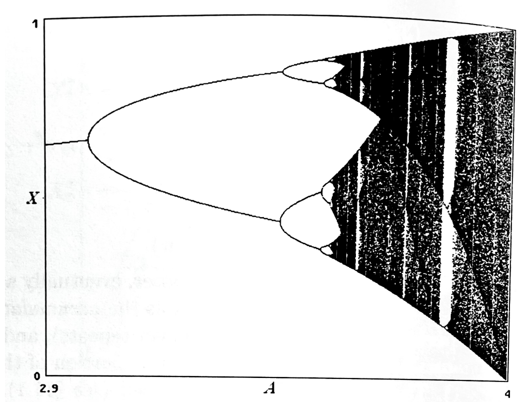


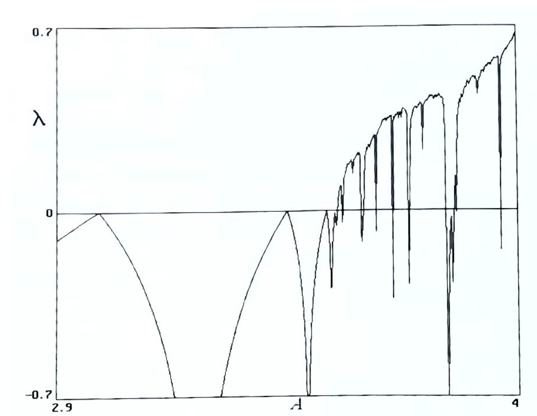
Screenshots of my double pendulum model in Mathematica

Now this system is chaotic, but how do we quantify the level of chaos here? The Lyapunov exponent uses the key definition of chaos, with the idea of a sensitive dependence on initial conditions, and measures if and how quickly two paths diverge. A positive Lyapunov exponent signifies chaotic behaviour, and the magnitude of this value shows how chaotic a system is. The Lyapunov exponent, for one-dimensional cases, is defined as follows:

$$\lambda = \lim_{t\to\infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} \tag{28}$$

Here δZ_0 represents the initial displacement between two starting points, which we want to tend to 0, showing how two initially very similar paths diverge. So then $\delta Z(t)$ represents the displacement after t units of time. This displacement is the displacement in phase space, a way of representing every possible state of a dynamical system. With a double pendulum, there are four unknowns, θ_1, θ_2,$\dot{\theta_1}$ and $\dot{\theta_2}$, so phase space would be 4-dimensional. The $\ln$ is used as this is meant to represent exponential growth rates, and this is averaged over t. This Lyapunov number is then averaged over many different starting conditions to get the global Lyapunov exponent. A positive Lyapunov exponent means the paths diverge exponentially fast, and therefore are chaotic. A single value for the Lyapunov exponent of a double pendulum cannot be given, as this varies greatly with smaller changes, such as the masses and lengths of the strings, but below is a graph showing the Lyapunov exponent for a logistic map, another chaotic system based upon a recurrence relation.[4]





Left: The logistic map for a range of constants, Right: The Lyapunov exponent for that same range of constants. from [5]

[3]If you want to have a go at playing around with a model of this, visit https://www.myphysicslab.com/pendulum/double-pendulum-en.html
[4]For more explanation and depth into this separate system, see [4]

Returning to the quotation in the introduction of this essay, often referred to as 'Laplace's Demon', the main idea of which claims the possibility of telling the future. As Stephen Hawking said in his book [6], "Laplace suggested that there should be a set of scientific laws that would allow us to predict everything that would happen in the universe", as we have found with the pendulums. A more subtle aspect of this suggests that nothing in the universe is random (i.e. non-deterministic).

However when we consider stochastic systems, the other type of dynamical system, randomness plays a significant role, and a good example of this is the quantum double pendulum, the quantum counterpart of the classical double pendulum we have been looking at. This is similar to the quantum billiards table, where particles don't have a definite location, but instead are randomly evolving with a probability functition that evolves solely deterministically by the laws of quantum mechanics. Significant conjectures are known about the allowed energies that these particles or billiards can have given the rules of quantum mechanics, that these levels, like electron orbitals, can only take discrete values, unlike in continuous systems, where any value of energy is possible. Amazingly, the spacings of these energy levels have links to their classical system. When the corresponding classical system is not chaotic, the energy levels are completely "uncorrelated", but with when the underlying system is chaotic, as in the double pendulum, the spacings of the energy levels are given by the spacings of eigenvalues of random matrices, as are the zeros of the Riemann Zeta function [7].

If the universe fits into these set of scientific laws, this suggests that quantum systems are also non-random, yet all the research into quantum mechanics has shown that these systems are inherently uncertain, meaning that, unfortunatly, unless a huge discovery is made in the world of quantum, we won't be able to predict the future any time soon.

So, even though chaos prevents us from predicting the future, as Laplace had assumed, simply because we cannot find these initial conditions, chaos theory is still fascinating and could potentially be used to understand some of the deepest problems in number theory, and to investigate some of the metaphysical ideas at the heart of our universe.

References

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[7] Z. Rudnick, "What is quantum chaos?," *Notices of the American Mathematics Society*, vol. 55, p. 32, Jan. 2008.