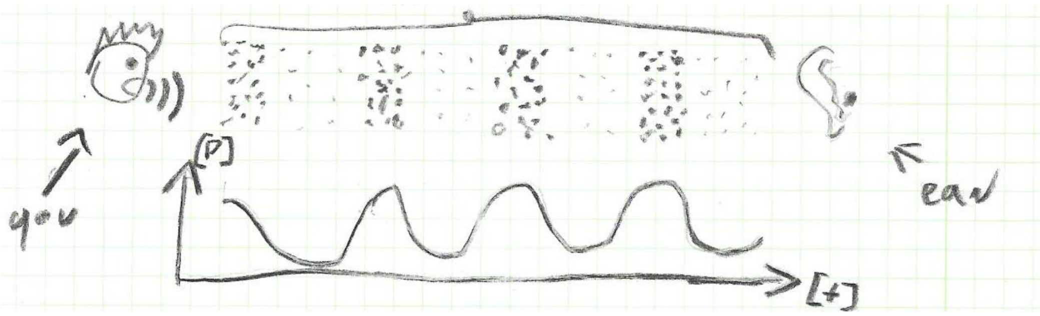
Why does the guitar sound different from the piano? What makes our voices sound different? Those are the questions I have asked myself for a long time. Not surprisingly, acoustics with the usage of mathematical models can answer this question very well. Unfortunately, the mathematics behind that is often very complex, but I will try my best to explain it as simply as possible. In this article, you will learn what Fourier transform is, and its application in analysis of sound timbre.

Some people, trying to answer my question would say *'Well, the piano and guitar have different timbres, that's why they sound different.'* and of course, this is the right answer. But what exactly characterizes the timbre of a piano or guitar? First, let me introduce you to some concepts.

As you might know, the sound is the changing pressure of air. When you open your mouth wide and force your vocal cords into vibration it will transmit those vibrations into the air, and they are nothing else but compaction and rarefaction of air particles over time. This is how it looks like:



In reality, the change of pressure over time is not a basic sine or cosine wave, it is a complex non-periodic function. Every voice or instrument has its unique "wave", which is the first reason those sounds are different. For example, in the figures below, you can compare the waves of the piano and the human voice.

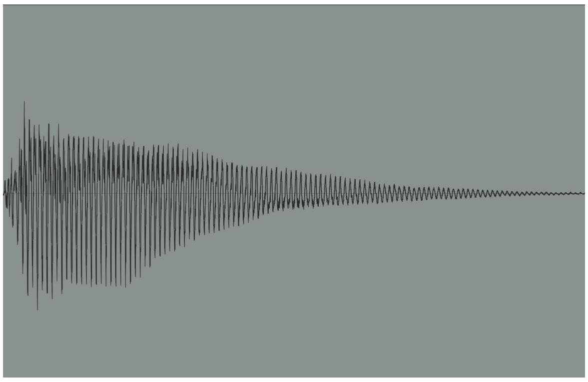


Figure 1 - wave of C4 played on piano.

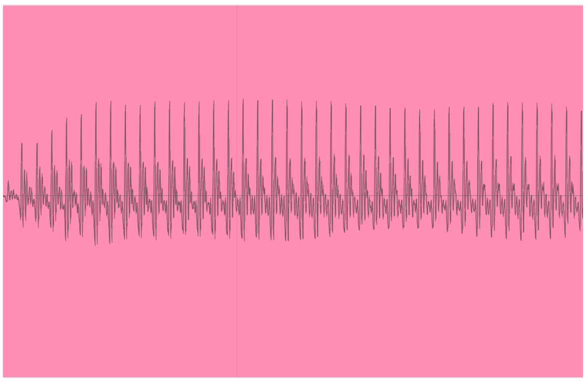


Figure 2 - wave of Human saying an "a" vowel.

As you may notice, those two waveforms are completely different. Though what makes them different? Of course, you could simply say *'Well, the shape is different'*, this is also a correct answer, however, let's have a deeper look at it, let's understand what is behind this shape.

Despite that, before we dive into this beautiful abstract, Allow me to introduce you to a pure tone. The pure tone is the simplest sound you can ever hear, with the simplest waveform you will ever see (silence does not count). It looks like this (Figure 3):

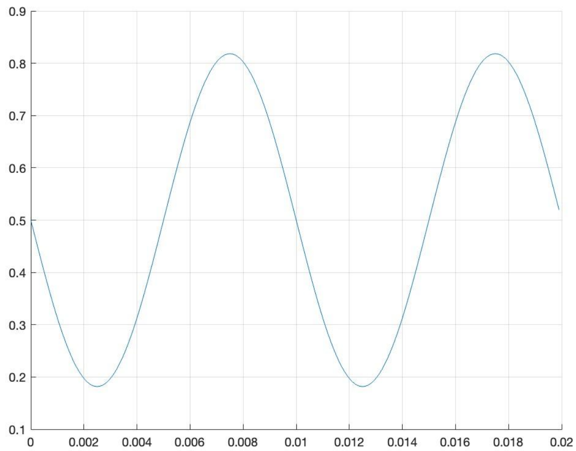


Figure 3 - Two periods of pure tone with frequency of 100 Hz

Yes, it is just a sine wave. As you might see, the sine depicted on the graph has a little phase shift, nevertheless, phase shift does not cause a difference in sound. For example, a cosine in acoustics would be called a sine with a 90° phase shift.

Allow me to demonstrate something, let's add two pure tones of different frequencies. One with a frequency of 100 Hz which is our base tone, and the second one whose frequency will be twice as big as the frequency of our base tone, so 200 Hz.

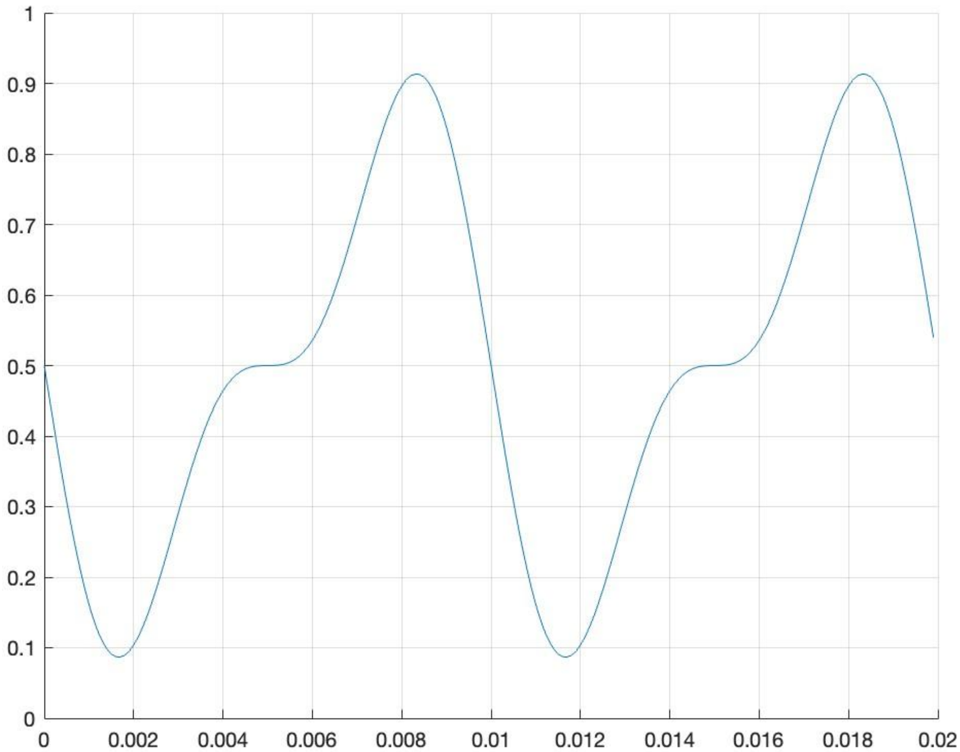


Figure 4 - Two periods of sum of waves 100 and 200 Hz

Looks interesting. Now let's repeat the process and add to our 100 Hz + 200 Hz wave a 300 Hz wave, so three times greater frequency.

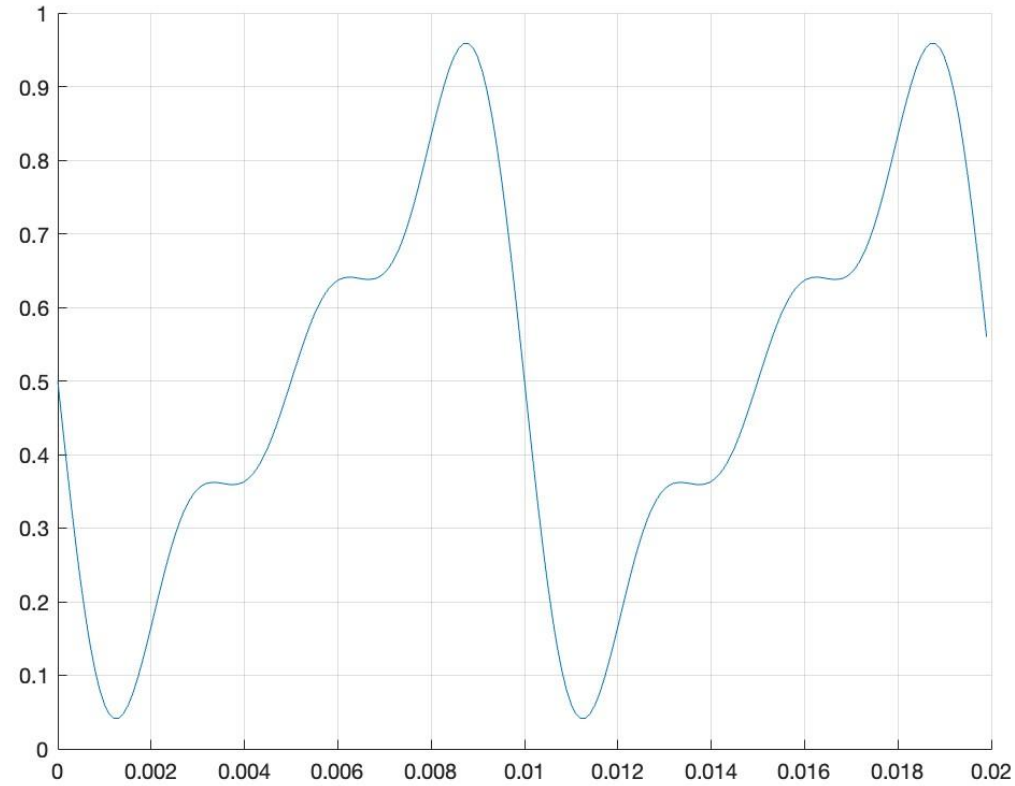


Figure 5 - Two periods of sum of waves 100 + 200 + 300 Hz

Also intriguing. Let's do something extreme right now. Let's add 100 sines in a way such that the first sine is our base tone with a frequency of 100 Hz, and the following terms are multiples of it, which we will call harmonics. As a mathematical expression, it looks like this:

$$w(t) = \sin(2\pi \cdot 100[Hz] \cdot t) + \sin(2\pi \cdot 200[Hz] \cdot t) + \sin(2\pi \cdot 300[Hz] \cdot t) + ... + \sin(2\pi \cdot 10000[Hz] \cdot t)$$

And this is its graph:

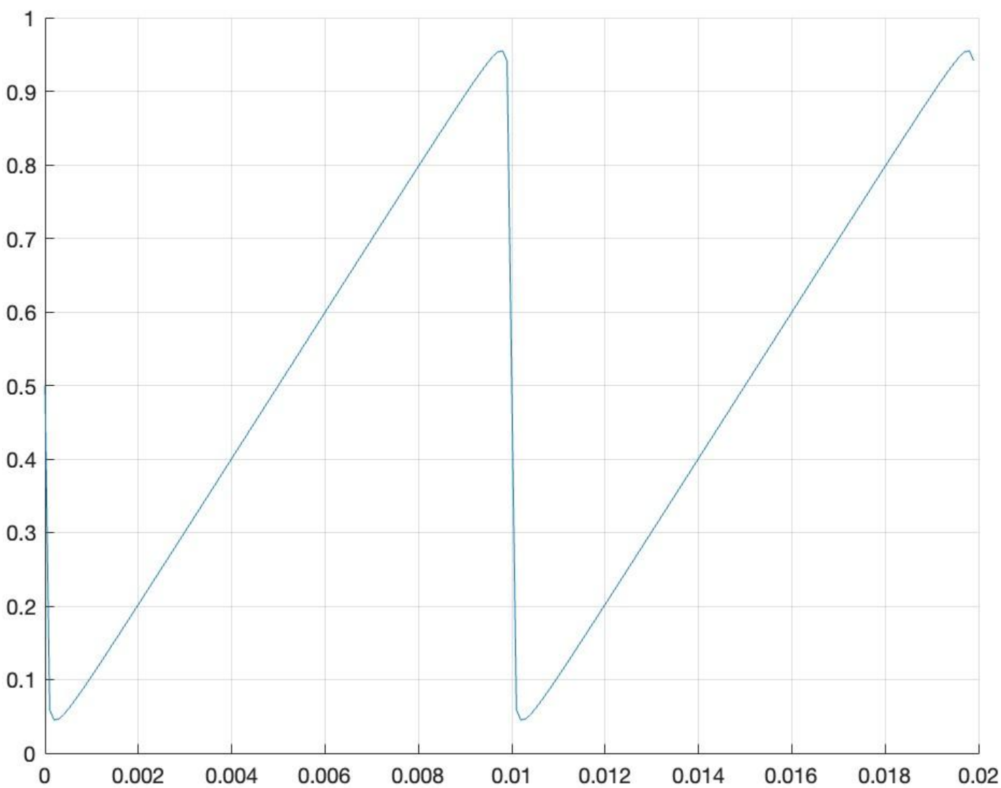


Figure 6 - Two periods of sum of hundred sines

The waveform you see above is called a sawtooth or saw wave (because the shape looks like a saw, isn't it?). The process of adding up sines with the method we did above is related to the mathematical construct called a Harmonic series. It can be expired in this way:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

The difference is that the harmonic series is an infinite sum, and in the real world we can not have an infinite amount of terms. However the more terms you add, the more accurate shape of a saw wave you will achieve, but as you probably see, 100 is perfectly enough. The second major point that we have to mention is that the harmonic series is related to wavelength, not frequency. But from the underneath equation, you can notice the relationship between frequency and wavelength. *(V here is equal to 340m/s because it is the speed of sound on earth.)*

$$\lambda = \frac{v}{f}$$

In more friendly words, twice greater the frequency, twice shorter the wavelength. As I mentioned above the frequency that we started with is the base frequency and the sines that are multiples of our base frequency are called harmonics. Despite that adding along all harmonics, is resulting in a sawtooth wave, we are not obligated to add up all terms. We could for example add only odd harmonics to our sum. In the figure below, you might see the result of such a process.

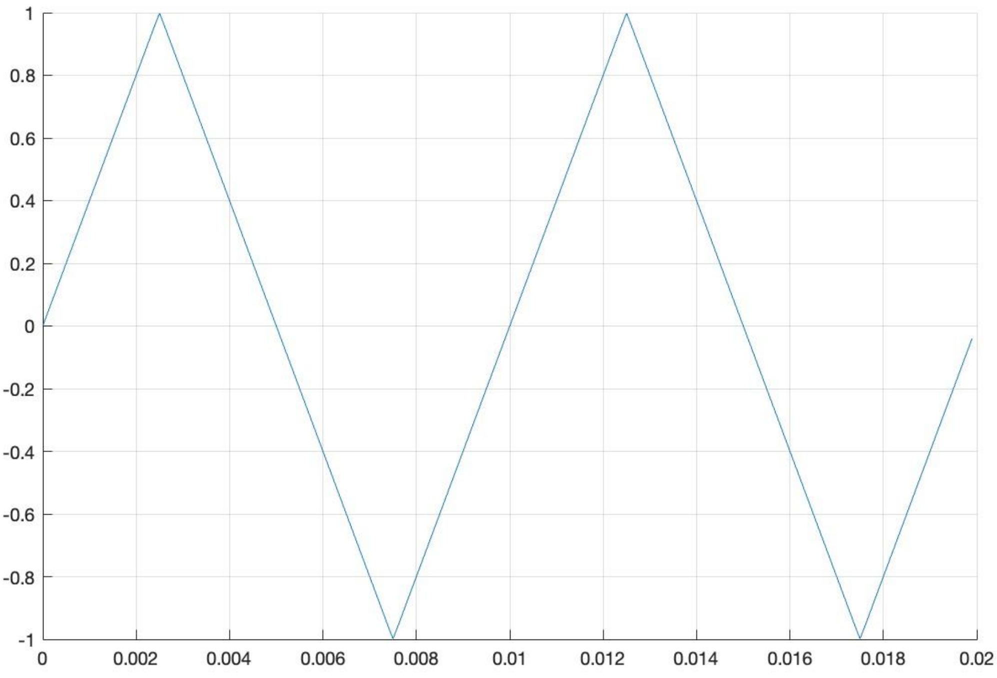


Figure 7 - Two periods of sum of even harmonics

All the waves we have discussed in the previous part are periodic waves. And as said in the very beginning, sounds like a piano or our voice is non-periodic waves, also you could notice that on the provided figures. You can ask yourself why I showed all of these. What if I told you that all existing sounds could be described as the sum of harmonics?

In 1821 Mathematician Joseph Fourier claimed that any function can be expanded into a series of sines. He created something called the Fourier transform which is a transform able to convert any function into a frequency domain. If you input any function to the Fourier transform, it will output the pure tones included in this function. For example, if we wanted to know all the harmonics included in the sawtooth wave (which we know already, but let's say we do not), and we input it into Fourier transform, this is the output we will be provided with.

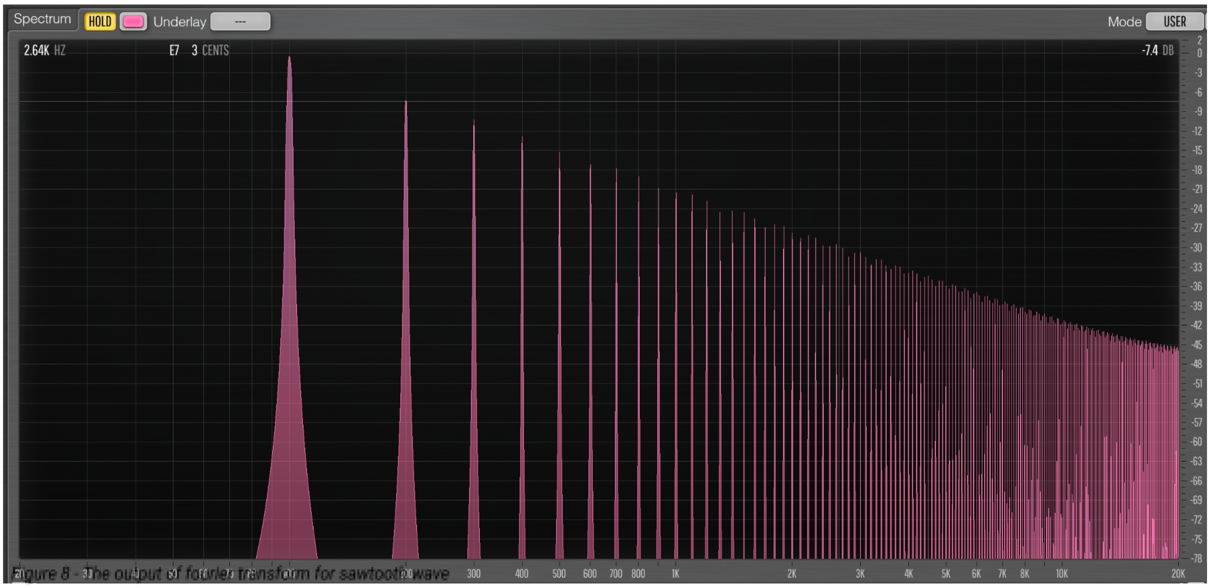


Figure 8 - The output of fourier transform for sawtooth wave

The Fourier transform is actually a consequence of the Fourier series which is actually what I showed you above, we created periodic functions out of sines. And I'm sorry, I lied to you a little at the beginning. It is not only about adding up the sines, but each sine has its own amplitude. This is how we can represent a sawtooth wave that I showed before with a Fourier series:

$$s(t) = \frac{-2}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \sin(2\pi i t)$$

The sounds that were mentioned in one of the first paragraphs could be also called functions. Let's say that we play a sound A4 on a guitar and it produces the waveform function that we will call $g(t)$. If we plug this function into a Fourier transform we would be able to see all the sines that stand behind this guitar timbre. We could also do the same to A4 played on the piano. Then we are able to compare these two outputs and see the physical difference between those two sounds.

The unfortunate thing is that unlike in periodic waveforms like saw waves, the harmonics included in sounds like a guitar or a piano do not have a constant amplitude. That means that the output of operations that we did above is not accurate. To see what really happens in a signal we would need a tool that could generate a Fourier transform in the current moment. Those tools are called spectrum analyzers. Not getting into very details, they use an algorithm based on the Fourier transform called fast Fourier transform (FFT). This algorithm will take a slice of the processed signal, then compute the output for the Fourier transform created from this slice, and then go to the next slice. It will keep changing the output of the Fourier transform, so we will be able to see how actually all harmonics behave.

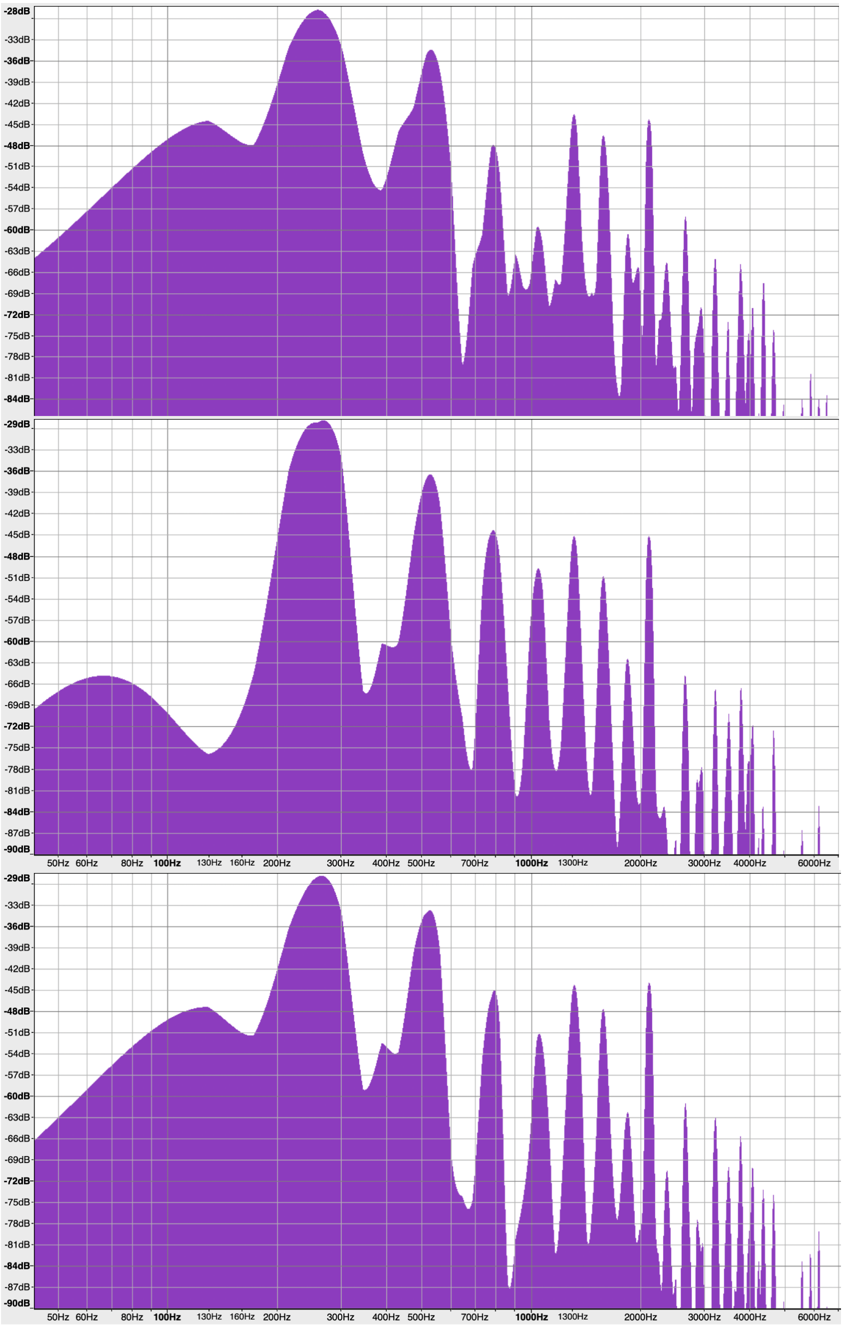


Figure 9 - The output of fourier transform for 3 slices of C4 played on piano

The truth is that the bigger slice we take, the more accurate the transform is, but it will take longer to change the output, so basically, we will not achieve the goal, which is to observe the change of harmonics over time. On the other hand, the shorter slice we take, the more inaccurate our transform is. This basically means that we never know how exactly harmonic's amplitudes change.

But let's consider a hypothetical situation where we know all harmonics of a signal and the change of their amplitude over time, then we could reconstruct that signal. For a sound of a piano, we could form such a function which is a sum of harmonics multiplied by amplitude which is also a function of time.

$$p(t) = a_1(t)\sin(2\pi f t) + a_2(t)\sin(2\pi 2 f t) + \ldots + a_n(t)\sin(2\pi n f t)$$

Finally, with harmonic analysis, we are able to see the physical difference in the timbre of the guitar and piano. The figures below are spectrums of a guitar and of a piano. As you probably guess, those graphs do not provide information about the change of amplitude, but generally, you will be able to see the difference.

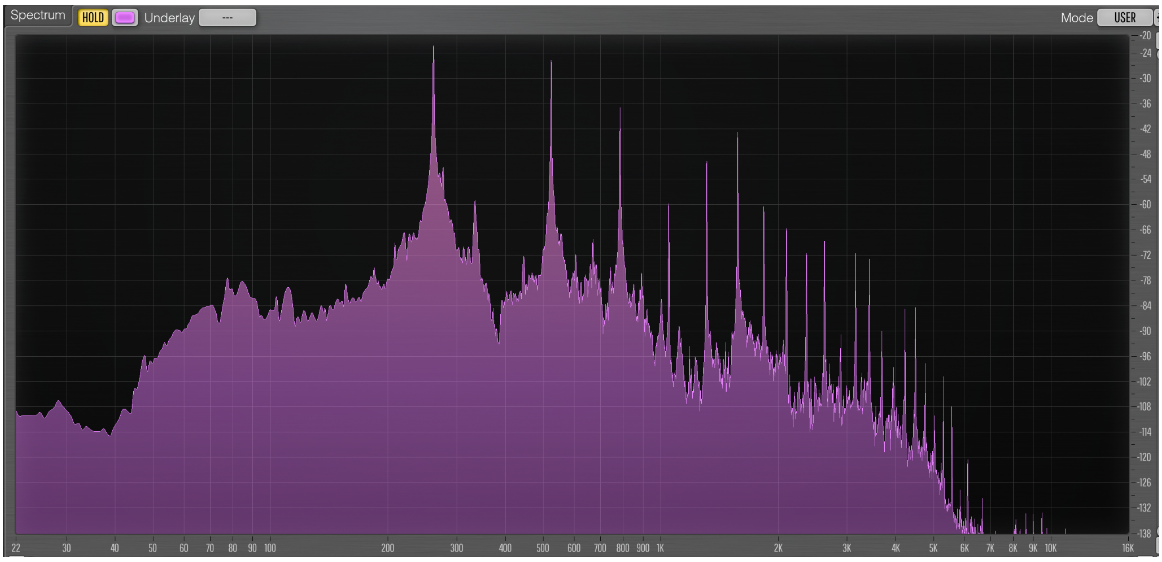


Figure 10 - The output of fourier transform for C4 played on guitar

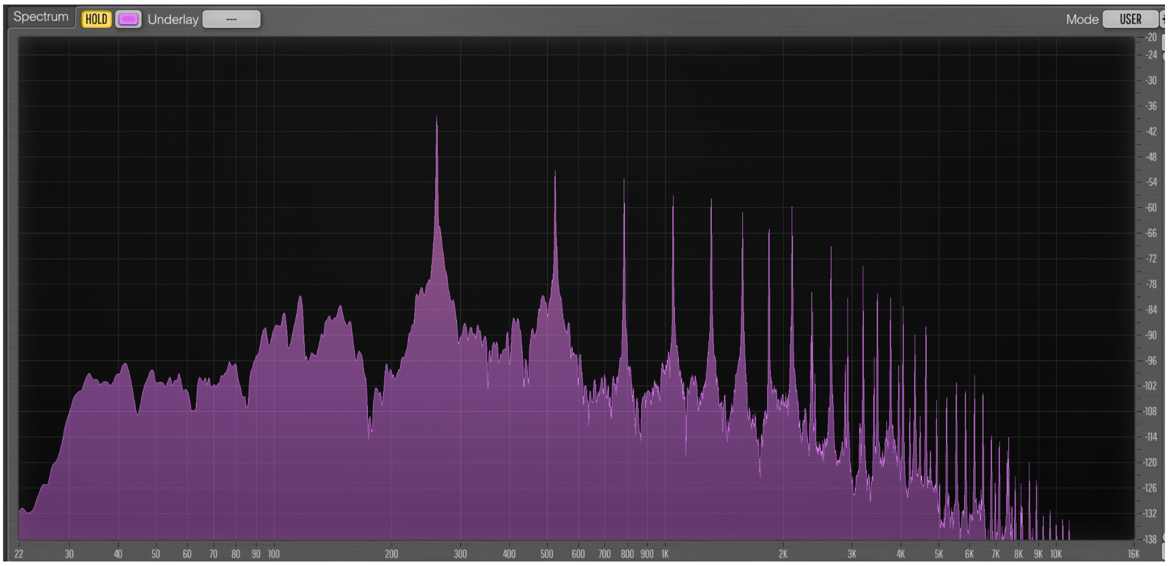


Figure 11 - The output of fourier transform for C4 played on piano

Sadly, we don't have enough time to deeply analyze those spectrums, but I leave you room for your own interpretation.

The analysis of harmonics (which I hope you will try to do on your own, with the signals above) is a very useful process, and it is applied in various aspects of electronics and acoustics. As you might already guessed, we could use it to recreate real sounds artificially. We can generate sines that we need, recreate their envelopes (change of amplitude over time), add them together and voilà, we have created the sound of a piano. Theoretically. This process is called additive synthesis and it is used in music industry to recreate sounds or sometimes in voice synthesis.

Some more advanced readers probably noticed, I have not provided any formula or equation related to the Fourier transform. The first reason why is that those formulas are very complex, and they need a certain mathematical background. The second reason is that I wanted you to understand the concept, how it works, and how it can be applied to pragmatic solutions. This article is missing a lot of details, and I know perfectly that if someone who is experienced in mathematics and/or acoustics would read this article, they will certainly notice that, even so sometimes it is much more important to understand the idea.