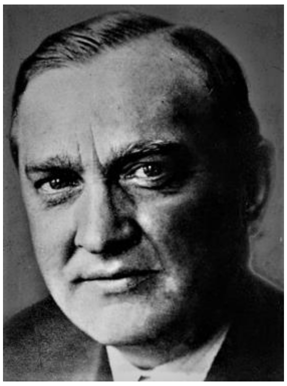
Maths: Fundamental Reality or Human Masterpiece?

Is maths “the alphabet with which God has written the universe” [Galileo], which is fundamental to reality, or as Stefan Banach puts it, a “powerful creation of the human spirit” – a tool that we use to better understand our world?

[Galileo Galilei]



[Stefan Banach]

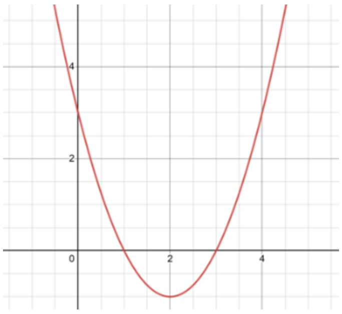
Let’s first consider this: is Macbeth fundamental to the universe? Is it one of those things which just exists, outside of human interference? Once the words are put on the page in the order Shakespeare arranged them, regardless of the language or symbols used (given that their meanings are equivalent), Macbeth is made. But it would be absurd to claim that Macbeth is fundamental to the universe, as it is a human’s idea inked onto paper from their thoughts. So why is it any different with mathematics?

Working through Macbeth in school, everyone developed different views of the text. Were the witches evil? Was Lady Macbeth in control of herself? Did Banquo actually have a ghost? These varying interpretations are caused by inconsistencies or a lack of clarity present because Macbeth is a human’s idea. Can the opposite be said for maths? Is maths a way of describing the universe, void of inconsistencies, and therefore intrinsically real?

Firstly, what is maths? Mathematics is the “abstract science of space, number and quantity” [Concise OED]. The question we set out to ask is, in essence, whether maths is a human invention or not. Ask yourself: “Did 2+2=4 before we defined those symbols?”

Throughout history, maths has expanded through a cycle of discovery, the construction of notation, and then proof by prior knowledge that the proposal holds true. This constructed the model which counts sheep, dictates the global economy, and sent us to the moon. Any new discovery is true to the system if proven using the system itself – maths is created by maths. Either it's all a construction or it's all fundamentally real. So where can we find our answer if the system is closed? There is one entrance, right at the start of the string of logical reasoning, the foundations of mathematics called the axioms.

All models are assembled from statements called axioms, declarations which are self-evidently true, like the periodic table in chemistry. Mathematics is both a model and a science (discussed later) and it's theorems are also constructed from axioms. It is these axioms which create a system of logical reasoning, allowing every mathematical statement to have a 'truth value' (be true or false). These are consistent no matter how you complete a problem (given that no errors are present). For example, you can solve $x^2-4x+3=0$ with the quadratic formula, factorisation or completing the square, all leading to the same solutions. This is why our axiomatic maths system is perfectly consistent.



[The graph of $y=x^2-4x+3$, $y=(x-3)(x-1)$ or $y=(x-2)^2-1$, Desmos]

Every proven mathematical idea is indisputable under mathematics itself, due to this consistency and the processes of logic. We will assess the two primary sets of axioms, and the consequent conclusions we acquire, assuming that they are true and that they complete the system of mathematics.

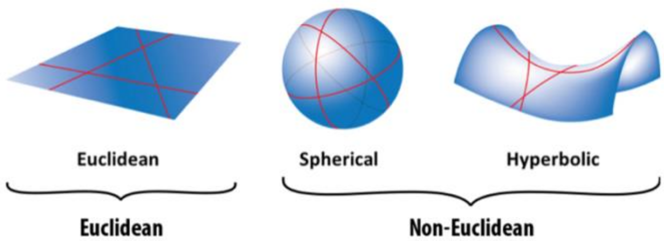
Geometry's axioms were defined by Euclid [Euclid's First Book of Elements]. Non-Euclidean geometry also exists by replacing the 5th postulate (below) with an alternative, but the existence is just a technicality which still relies on Euclid's axioms. Euclid stated the following 5 postulates (axioms):

1. A straight line segment can be drawn to join any 2 points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. If two straight lines segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough. (Non-Euclidean geometry replaces straight lines with different curves).

These axioms cover all of geometry, with each geometrical theorem provable using them as first principles. If geometry is intrinsically true, then so are these axioms.

So are the axioms true in the real world? Picture two points on a football pitch; there is an obvious straight line route from one to the other. Picture two planets; a neutrino could go directly from one to the other, and then carry on indefinitely. It is clear that each postulate carries indisputable truth in the world we observe, so each axiom can be viewed as a real statement. Therefore, everything within geometry is fundamentally real within our universe if proven using the axioms as the only starting points.

[Euclid]



[Non-Euclidean Patternmaking, Dr Mark Lui]

Geometry is easy to visualise as it isn't an abstract concept. We can draw diagrams, measure length and area and make observations from physical patterns in the world. Although we can't envision the concepts, assessing pure mathematics leads to the same conclusion.

Most other fields of mathematics share the same set of axioms. Whilst a couple of others exist for concepts like probability and set theory, most fields within pure mathematics, and all of their applications, rely on a short list of axioms which seem far too trivial to lead such a long chain of complexities. The notation of an axiom is irrelevant, it is the idea it encapsulates which is important. The list has the following 15 axioms:

1. $0 \neq 1$
2. $a+b=b+c$
3. $a+(b+c)=(a+b)+c$
4. $a+0=a$
5. $a+(-a)=0$
6. $a \cdot b=b \cdot a$
7. $a \cdot (b \cdot c)=(a \cdot b) \cdot c$
8. $a \cdot 1=a$
9. $a \cdot (1/a)=1$ (if $a \neq 0$)
10. $a \cdot (b+c)=a \cdot b+a \cdot c$
11. If $a,b \in \mathbb{R}$, then one and only one of the following is true $a>b$, $a=b$, $b>a$
12. If $a,b,c \in \mathbb{R}$ and $a>b$, $b>c$, then $a>c$

13. If a,b,c∈ℝ and a>b, then a+c>b+c
14. If a,b,c∈ℝ and a>b, c>0, then a·c>b·c
15. Every nonempty set of real numbers that is bounded above has a least upper bound.

1-10 are the basic axioms of mathematics, 1-14 are the 'order axioms', 15 is the 'completeness axiom'.

Whilst this list initially looks complicated, and makes maths seem like a very theoretical concept, consider what each axiom states; they're called the basic axioms for a reason.

Like with geometry, if these axioms are intrinsic truths in the real world, then so is the maths that follows. 0≠1 is difficult to contest; if I have an apple, I don't have no apples. If I have 1 pen and buy 2 pens, I finish with the same number of pens than I would if I had 2 pens and bought 1 (∴ a+b=b+a has real world truth). Continue this process and you find that each axiom's idea is indisputably true in the real world. The same conclusion is drawn when we assess other mathematical fields' axioms as they are equally trivial and equally grounded in the real world. Therefore, the axioms' intrinsic truth within our universe implies that all maths which follows is fundamentally real as well, concluding that all maths has inherent reality.

Whilst this conclusion is true under our assumptions, Kurt Gödel exposed a potential flaw in 1931: the axioms do not complete the system of mathematics.

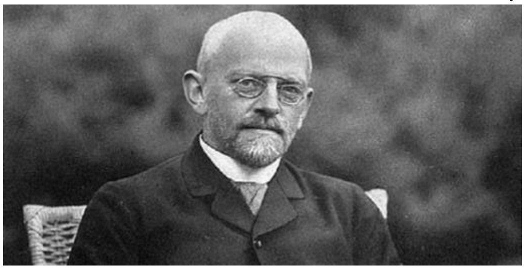
[Kurt Gödel]

Gödel's 1st Incompleteness Theorem [Theorem VI, On Formally Undecidable Propositions of Principal Mathematica and Related Systems I] uses the same self-referencing property of verbal paradoxes, such as "this statement is false", to display incompleteness within our axiomatic system of maths. As mentioned earlier, sentences are a human construct with no fixed truth values, so these paradoxes are to be expected. Gödel believed a similar approach could be used within maths if he could allow numbers to talk about themselves.

He constructed a system called 'Gödel coding', where every mathematical statement could be converted into a unique number. This brought every mathematical claim together, where statements had codes based on which other statements had proved them or disproved them. The axioms had numbers, and if these numbers were combined to form a proof, or disproof, you had the code for the proven/disproven statement. This allowed Gödel to construct a sort of proof by contradiction when he gave a value to his proposition: “This statement isn’t provable by the axioms”.

Since the statement is mathematical, only discussing proof and the axioms, it must be provable or disprovable under Gödel coding. If we assume that the statement is false, that makes it “provable by the axioms”, meaning it is true; so the statement must be true as it being false is self-contradictory. The fact that it is true generates a ‘mathematically proven unprovable statement’ – not all maths is provable. This means that the axioms are an incomplete list since they don’t allow all of maths to be proven.

This upset some mathematicians as it created the potential that their work on certain proofs may be futile. David Hilbert attempted to resolve this by allowing Gödel’s statements to be considered axioms, but this still allowed similar statements to form indefinitely. All of mathematics could therefore never be proven with a set list of fundamental axioms.



[David Hilbert]

So the axioms create an inherently real system, which we see the applications of every day, but they’re incomplete and mathematics can never be complete. What does this mean?

Maths can’t be used to understand all mathematical ideas. With any scientific model, this is a major issue; if our model for ionic bonding couldn’t portray calcium carbonate, then it would be indisputably flawed, and flawed concepts should not be considered to be real. This means that if maths were to be a model, we could conclude that it is a flawed human concept. However if it is a science (as it is defined), then we can conclude that it is fundamentally real. For example, physics is a science which studies natural phenomena, so its concepts are appreciated as real even though we may never be able to truly understand subatomic particles in the non-quantum world.

Is maths a model? Models are constructions, which are used to simplify concepts to aid human understanding. Maths is a simplified representation of what we can observe, providing abstract concepts in a system which are then used to understand real-world phenomena. It is with maths that we generate theoretical ideas within physics and economics to make predictions about things we cannot observe. So maths is a model.

Is maths a science? Sciences are studies of the natural world, which generate theories by observation, and then use logical study of patterns to confirm or deny these theories. Maths takes observations of quantities, patterns, and shapes and then generates theories to try and comprehend these observations. It then uses its axioms, which are observably real in the world, to logically confirm or deny the theories, and generate theorems. Every mathematical development is also peer-reviewed to sustain consistency and truth. So maths is a science.

As maths is both a model and a science, it follows that it is both manmade and fundamentally real. How is this possible? As we know from the axioms, maths is rooted in truth, so all true ideas in maths are real even if we cannot confirm them all with proof. However we define maths, since it is an abstract science. This means that we cannot study mathematical phenomena without human intervention, so all maths which we encounter is a universal truth which has been formalised by people.

By defining our question, considering the process of mathematical development, studying the axioms, unpacking the significance of Gödel's 1st Incompleteness Theorem, and then showing the dual nature of mathematics, we have concluded that there is inherent reality within maths, regardless of how we denote it. However because of maths' abstract nature, it cannot exist without human creation. Therefore, maths is simultaneously 'God's alphabet' and a "creation of the human spirit", and to claim one without the other would deprive the field of its intrinsic complexities.

Jack Pennystan, 2023