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About the beautiful mathematics behind stacked chips

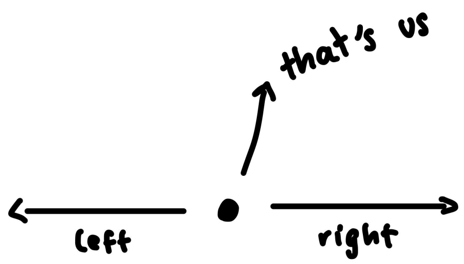
An essay on dimensions, coordinates, functions and the beauty of mathematics

Prologue

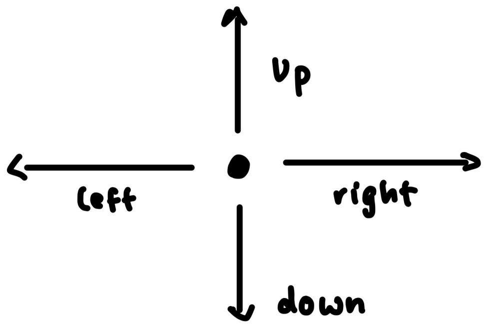
Dr. Tommy, a great mathematician exploring the world of fluid dynamics, and his good friend Jimmy, a famous painter and sculptor, are taking a walk, on a beautiful Sunday, through the oxford university parks. After walking for some minutes, they sit down on a bench and begin to talk about pokémon, art and the origin of life. While the two chat comfortably, Dr. Tommy begins to open a can of stacked chips. “Why are you staring at your chip?” asks Jimmy with his mouth full of chips. "The shape…" murmurs Dr. Tommy to himself.
Jimmy is confused about what Dr. Tommy is talking about. Dr. Tommy is enthusiastic about the shape they are looking at, which he considers to be pure, beautiful math. When Jimmy asks for clarification, Dr. Tommy explains that the chip they are examining is a hyperbolic paraboloid. However, Jimmy still doesn't understand and asks for further explanation.
“Close your eyes Jimmy, let’s explore the beautiful world of mathematics…”, says Dr. Tommy with a smile on his face.

The Cartesian coordinate system

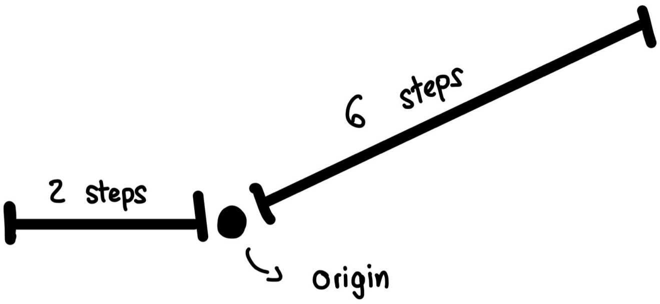
Imagine a little dot, that's us. We are a one-dimensional creature, living in a one-dimensional world.
In geometry, a one-dimensional system is often represented by lines, which can be thought of as a set of points that extend infinitely in both directions. So one-dimension refers to a single direction or axis in space. In short; we can move left or right.



But that would be a bit boring, wouldn't it? It would be much cooler if we could also move up and down. So let's just change our one-dimensional world to a two dimensional world. Two dimensions refer to a system or space that has two measurable parameters or directions, commonly known as length and width. In geometry, a two-dimensional object or shape can be visualized on a flat surface, such as a piece of paper, where it has only length and width, but no depth. Examples of two-dimensional shapes include squares, circles, triangles, or our two-dimensional world right here.



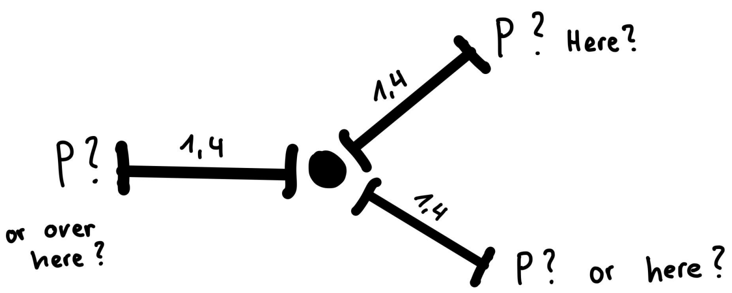
Now, we are able to move freely in all directions which are possible in our system. We can move from top to bottom, or from the top-left corner to the bottom-right corner, and if we colorize our paths, we can even draw all kinds of two-dimensional shapes. That's a lot of fun, isn't it? But full of joy, it can happen that we get lost in our big wide world. What happens if we get lost? How do we know where we are? To solve this problem we introduce a reference point. We call this point "origin". By introducing this point we also introduce the concept of distances.This helps us to determine exactly where we are within our system.



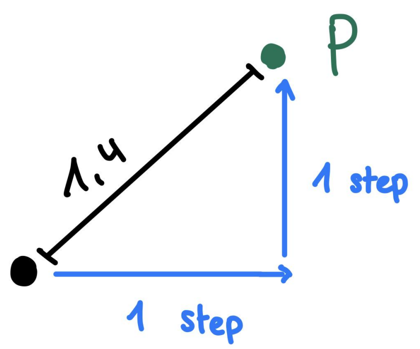
For every place in our system we now know, if we count our steps, how far away we are from the origin and hence where we are in our two-dimensional world.

But there is still a problem.

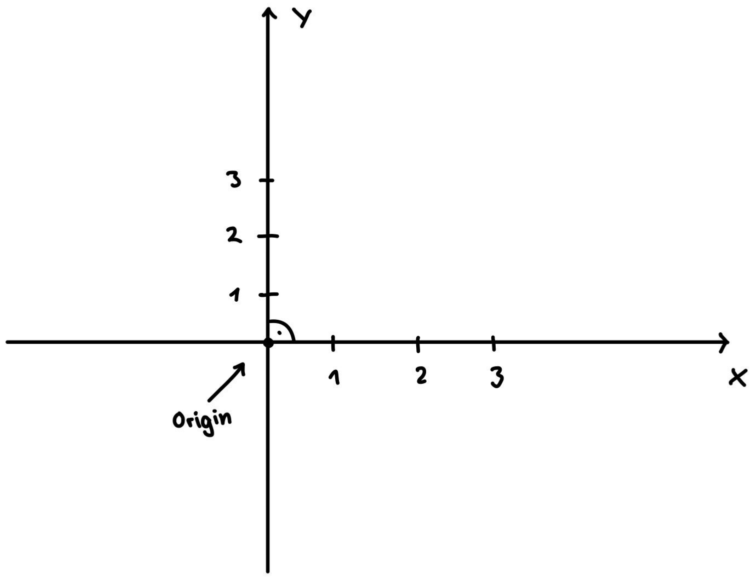
Suppose a friend lives in place P and the only information we have about this place is that P is about 1,4 steps away from the origin.



Based on our information the place could be on any point which has a distance of 1,4 steps to the origin. So we kind of know where the place P is, but actually we don't know the exact position. That’s the reason why we need some other kind of system to describe our position in our two-dimensional world, instead of just saying that our position is 1,4 steps away from the origin.
Suppose the place marked in green is the place P where our friend's house is located.
Now we have to ask ourselves, how can we get to this point? Exactly, we can reach this point by walking 1 step to the right and 1 step up.



With this method we can reach every imaginable place in our system. But we can make our life even easier by introducing 2 perpendicular axes. The horizontal axis is called the x-axis, while the vertical axis is called the y-axis. We label the axes with numbers, which indicate how many steps it took to get to a certain point on an axis.



Furthermore, we can now give every point in our system an exact address, by specifying the amount of steps we've taken in x-direction and y-direction. Our friend's house now has the following address:

$$P(1, 1)$$

We call this address a coordinate. Coordinates are a set of numerical values that are used to locate a point or object in a system. To define the position of a point on a 2D plane, we use two numbers, called the x-coordinate and the y-coordinate. The x-coordinate tells us where the point is in the horizontal, left-right direction, and the y-coordinate tells us where it is in the vertical, up-down direction.

In our example the coordinates of the point P are 1 and 1. The first (x) coordinate tells how far along the horizontal (x) axis it is located, here 1. The second (y) coordinate tells how far up the vertical (y) axis it is located, here also 1.

Congratulations!

We built a Cartesian coordinate system together. The Cartesian coordinate system is a fundamental concept in mathematics that allows for the representation of points in a plane using coordinates. The system consists of two perpendicular lines, known as axes, that intersect at a point called the origin. The horizontal axis is labeled x and the vertical axis is labeled y. Points on the plane are located by specifying their x and y coordinates, which indicate their distances from the origin along each axis. The Cartesian coordinate system is named after the French mathematician René Descartes, who developed the system in the 17th century.

Functions (the parabola & the hyperbola)

By using many dots, and connecting them, you can represent all kinds of different shapes.

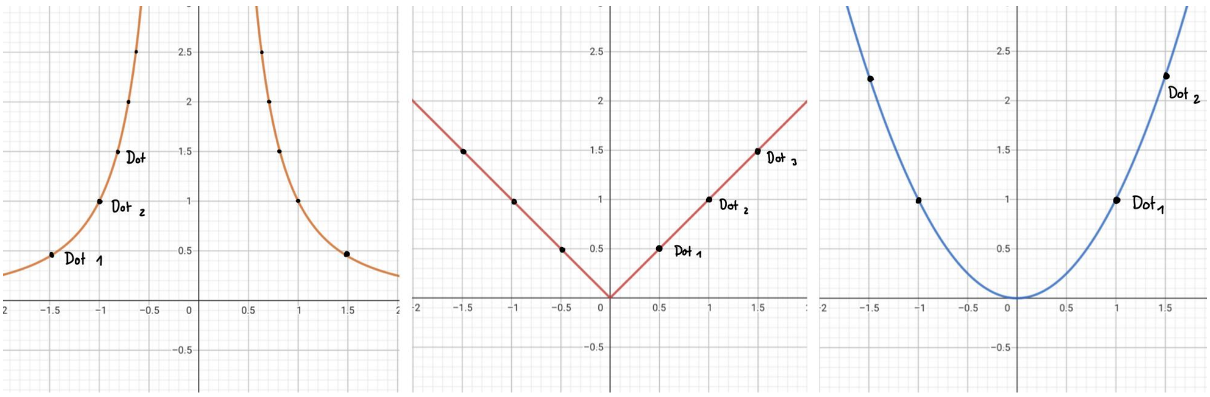
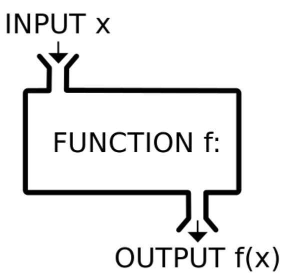


Figure 3 looks really cool, right? It would be really nice if there was some kind of blueprint that exactly describes how to build such a shape, wouldn't it?

In fact, there is such a blueprint! We call this blueprint a function. A function $\mathbf{f}$ is a relationship between two sets of values, called the domain $\mathbf{D}$ and the codomain $\mathbf{Z}$, such that each element $\mathbf{x}$ of the domain is uniquely associated with an element $\mathbf{y}$ of the codomain.



$$f : \begin{cases} D \to Z \\ x \to y \end{cases}$$

This means that for every input value of the function, there is exactly one output value. For the element of the codomain associated with the element:

$$x \in D$$

one generally writes:

$$f(x).$$

Functions are often represented using equations or formulas that specify how the input values are transformed into the corresponding output values. The graph of a function, what we have called shape so far, is created by plotting points that correspond to the input-output pairs of the function. The input values are typically plotted on the horizontal axis (x-axis), while the output values are plotted on the vertical axis (y-axis). Each point on the graph represents a specific input-output pair of the function. By using this information we can find out how the function of figure 3 looks like.

Let's look at some input-output pairs of the graph:

x	y
-8	64
-7	49
-6	36
-5	25
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64

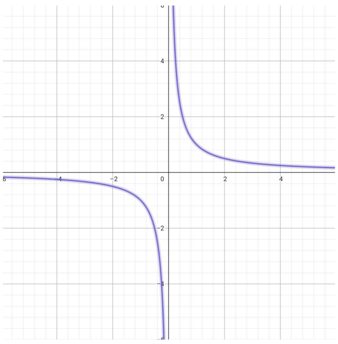
Using the table above, we can see that if an **x** ($x \in D$) is squared, the corresponding y value comes out. The function of the graph is therefore:

$$f(x) = x^2$$

The graph corresponding to this function is called parabola. There are also other functions which are linked to interesting looking graphs. The graph of the function:

$$f(x) = \frac{1}{x}$$

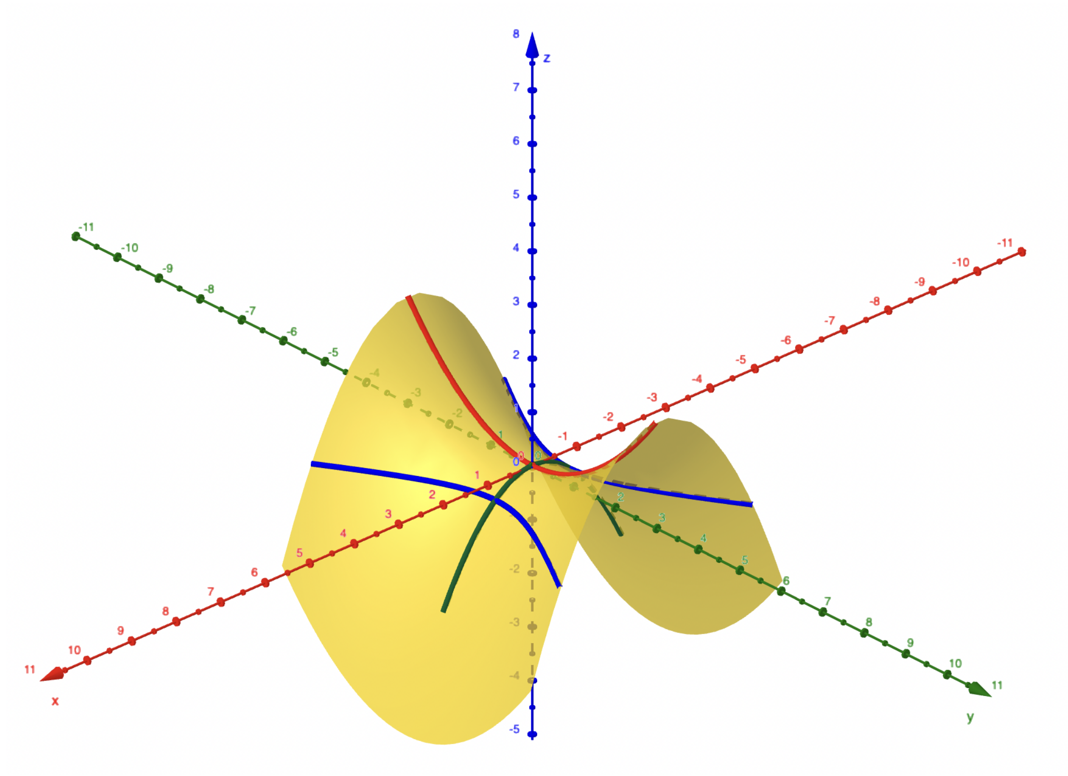
for example, looks like this:



and is called hyperbola.

Hyperbolic paraboloid

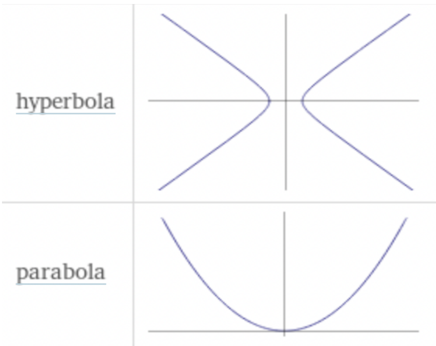
That's pretty cool right? But do you know what's cooler than two-dimensions? Correct! Three Dimensions! Three dimensions, often denoted as 3D, refer to a space that has three dimensions: length, width, and height. In such a space, objects can be positioned according to their x, y, and an additional z coordinate. This means that instead of 2 axes, we are now dealing with 3 axes. Instead of combining one-dimensional dots together to form some cool shapes, we now have the ability to combine two-dimensional shapes to form a three-dimensional shape. If we look at the following 3-dimensional body, we recognize some two-dimensional shapes we covered earlier in this essay. Namely 2 parabolas (red & green) and a hyperbola (blue).



This structure is given by the equation:

$$z = x^2 - y^2$$

Where x, y, and z are the three coordinates in three-dimensional space. This equation describes a surface that is doubly curved, meaning that this structure curves in two different directions, with one set of curves that are hyperbolic in shape (colored in blue in the image above) and another set of curves that are parabolic in shape (colored in red & green in the image above). The hyperbolic curves run in one direction (up and down), while the parabolic curves run in the perpendicular direction. The point (0, 0, 0) is the lowest point on the surface, where the two curves intersect, and the surface extends upwards and outwards from there. The surface of this structure has a hyperbolic cross-section in one direction (if you look at the structure from above) and a parabolic cross-section in the other direction (if you look at the structure from the side), which gives it its name: hyperbolic paraboloid. It is shaped like a saddle or a Pringles chip.



Hyperbolic paraboloids have a number of interesting mathematical properties and are used in architecture and engineering because they can span large distances without needing additional support. For example, a hyperbolic paraboloid can be used to create a thin-shell roof that covers a large area without the need for internal columns or supports. This is because the hyperbolic paraboloid is a doubly curved surface that distributes loads evenly across its surface, allowing it to support its own weight and any additional loads placed on it.

I consider the hyperbolic paraboloid to be beautiful because it is made out of other mathematical structures, like the parabola or the hyperbola and depending on the configuration also lines etc… Like Euler's identity is considered beautiful because it unites some of the most famous mathematical constants. For me, the thought that the entirety of dots, shapes etc… can achieve something higher dimensional is a beautiful one. We as humans should learn something from that, but that's another topic.

Epilogue

"And that is the beautiful math behind stacked chips, what do you think about this Jimmy?" asks Dr. Tommy.

dead silence

"Jimmy?! - are you even listening?"

snoring

References

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Images are from wikipedia.org, GeoGebra and WolframAlpha