Laura's Search beyond the land of plurality

As she was making her way to her first election, Laura couldn't help but be discouraged. Like many other young people she felt like the progressive candidate represented her views much better than any of the other two. However, she had no choice but to vote for the moderate candidate, otherwise the extreme right politician could prevail. It was all due to the plurality[1] voting rule used in her home country and the fact that pre-election polls said that the moderate and the far-right will go head to head for first place. "Oh, I wish we'd come up with a better voting rule than plurality" - said she, as she pressed the stamp a bit too violently beside the moderate candidate's name.

Suddenly, the lights dimmed, the calm, but still annoying retro song playing in the voting station stopped, as did the distant sound of the parties' observers quarrel, which seemed impossible to silence just a minute ago. Nothing around Laura was moving, it felt like the Earth wasn't either.

After a second of perpetuity, Laura heard a repetitive approaching clicking sound. "Tok-tok, tok-tok" - it was the sound of heels. One more second of perpetuity and in front of Laura was a middle-aged man dressed closer to the Middle Ages than to 2023. The previously annoying retro song would feel innovative in his presence. White, curly hair, resembling a toupee, puffy pants, heels and the rest of the 18th century couture package.

- Marquis de Condorcet - the man presented himself with an accompanying stereotypical bow - I appear in front of everyone dishing on plurality. Not to spook you, of course, plurality is not a religion. Quite the opposite, let me show you, that plurality is not the only way to go.

With a melodic combination of leg movements, resembling a Broadway Musical tap dance, the Frenchman transported himself and Laura into a large circular room with a spiraling walkway going all the way to the roof, around 15m above ground. All the walls were filled with packed bookcases and along the spiraling walkway stood human-size statues. "Am I the next Avatar?" - Laura thought for a second.

- Take a walk with me, - said the self-proclaimed Marquis.

They walked towards a table, which Laura thought looked like a DJ set. She didn't share this observation with the Marquis, he would hardly understand the analogy. The table, which looked fashionable only in the eyes of Condorcet and his contemporaries, had five or so levers, with words and Greek letters

[1] For those unfamiliar, the plurality rule in elections simply selects the candidate with most votes (not! a majority, i.e. $> 50\%$ of votes though).

along each lever, and a couple of other buttons. Laura was not just a nerd, but a math-loving nerd, i.e. the nerd supremum, so the Greek alpha, beta and gamma were obviously familiar to her.

- Alright, I know you have no idea what these are, but select a couple and let's see what happens.

- Emmm, ooke. - as Laura knew this is not really happening, she went along with it.

She selected γ for the lever called *Expansion*, *majoritarian* for *Granularity* and *Pareto-optimality* for *Monotonicity*. For diversity, she turned the lever called *Contraction* off.

- Ha-ha - the Marquis laughed in a very high voice - great intuition.

He showed her the big button with the text *Profit* on it and said smilingly as if asking for appreciation:

- This is a little joke for your generation, huh?

He proceeded to press the button. A statue somewhere in the middle of the spiral starting glowing with a bright blue. They went towards it. The statue read:

Nicholas Miller, Uncovered Set
"Select all uncovered candidates"

in a very squiggly font.

The Marquis showed towards it with the same "looking for appreciation" face.

- Quite a good idea my friend Nikola came up with, huh?

Laura's face already said everything, but she still enunciated:

- Mhm, just... one... question..., what are you talking about?

- Oh, right, I always forget. As my contemporaries used to say, let's start from the beginning and break this all down.

Tap dance - and, as if after leafing through a book, the two teleported to a large room, with fancy isle sitting, which looked like a parliament or city council.

- Remember you were saying a minute ago: "I wish there was some other way but plurality, bla, bla". You see, I have nothing to do in the eternal void, considering, ..., I criticised the wrong Constitution - he laughed awkwardly - aaand it's also been a couple hundred years, sooo ... in all scenarios ... dead - he said while pointing at himself.

- So, I pull people out here for 10 minutes or so, to explain to them the actual breadth of their choice.

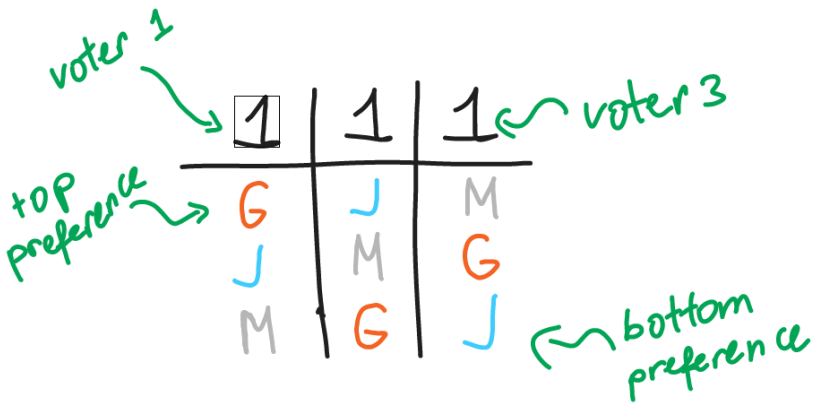
- As soon as people started voting more massively, some of them, of course, considered what should be the rules of that - what's the best way to select one group or another by means of a vote. And that's also what I and some of my contemporaries did, when we, you know, ended the monarchy - the Marquis tilted his head and clapped his hands in a slicing notion, obviously alluding to the guillotine - I was against that... process, by the way.

- And one of the things I understood and exemplified - said he, while twisting his hair with his fingers - is that voting is hard. I came up with this simple paradox, which people now call with my name: the Condorcet - he pointed at

himself smilingly - Paradox. Sounds quite good I would say. Many people say it is also the starting point of the science of social choice.

- Mhm, mhm, sure - Laura responded sceptically - so the paradox is?

- Well, it's rather simple. Imagine 3 people choosing between 3 candidates - a Girondin, a Jacobin and a Montagnard. The first one prefers the Girondin to the Jacobin to the Montagnard, the second - the Jacobin to the Montagnard to the Girondin, and the third the Montagnard to the Girondin to the Jacobin. If we write this down it looks like this. - he grabbed chalk from thin air and wrote down a table on a newly appeared board.

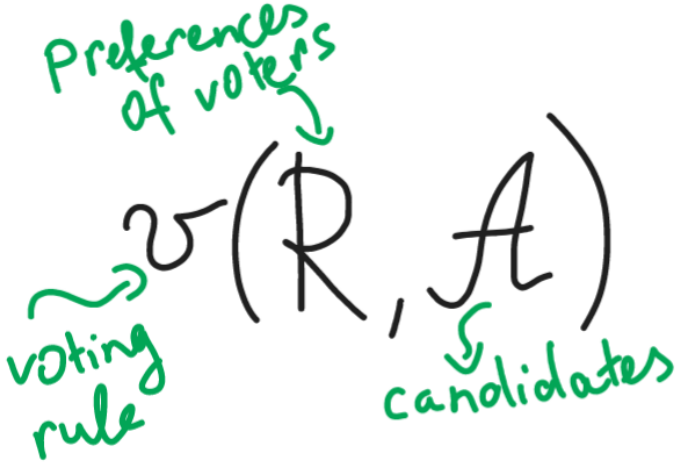


- And the paradox is quite simple, you cannot choose select one or two candidates in this situation as the result will be in some way unfair, either a voter would be discriminated against or a candidate.

- Oh, nice one, I like it - Laura nodded approvingly, expanding the Marquis' smile by a couple of millimeters - And this Nicholas Miller solved the issue with your paradox, or?

- No, no, let's go step by step. First, the basics. The science of social choice has very simple building blocks, we have SETS of candidates, which are called *candidates* and a group of voters. Each voter has an associated tuple of candidates, which represents their preferences amongst these candidates, going back to my paradox: the first voter has an associated tuple $(Girondin, Jacobin, Montagnard)$.

- This group of voters and their preferences build what's called a preference profile, the table I drew before. And now we can simply formulate the topic of social choice - what's the fair function, which given a set of candidates and a preference profile returns a winner candidate, or a set of such winner candidates or even a tuple of overall societal preferences amongst those candidates. And besides, what does "fair" mean in a mathematical sense?



- Are you planning to get more specific soon or should I try to wake up?

- Like is it fair to call someone a traitor and imprison them just because they think your constitution is worse than theirs, huh?

Laura looked at him puzzledly.

- Montagnard bastards... Sorry, lost my way - he smiled at Laura as if nothing happened - yes, of course, let's get more specific. So, to say that a function is "fair" in some sense, it needs to satisfy some properties. Remember the levers and buttons you chose from in my cinematographer?

- Cinematographer?

- Well, it's an interactive room, everything moves, "kinemat" in Greek is movement, I think it's a good term.

- Yeah, about that...

- So, this room. - "Tok-tok", book leafing, they are back in the "cinematographer".

- What you chose from here, are properties you wanted to be satisfied in your rule. Let's go one by one.

- *Expansion* properties deal with selecting the outcome of the function from a larger set of candidates, based on outcomes for smaller sets. So let's say there is a group of voters with some preferences, and in one election they can choose between Girondins (a), Jacobins (b) and Montagnards(c) and they choose a and b.

- Only not the Montagnards - he whispered under his nose.

In another election the same voters choose among a, c and d - the Thermidorians - and they choose a. Well, then it only makes sense that in the election among a, b, c, d with the same voters - the voting function has to select a. In a more general sense, with our voting rule v, the preference profile (i.e. voters' preferences) R and sets of candidates A and B, we have:

$$v(R, A) \cap v(R, B) \subseteq v(R, A \cup B) \tag{1}$$

This type of expansion consistency is exactly γ, the one you chose. There are stricter ones than γ, but those are more difficult to satisfy, of course.

- Now the other type of property you decided to leave blank...

- I just pressed some buttons, man...

- So, these properties deal with *contraction* consistency, meaning choosing something from a larger set should tell you something about choosing from a smaller set. Let's take α for example, to not complicate our squiggling further. The α property says - if you choose some candidate from a set, then it must be chosen in every subset of that set that contains it.

$$B \subseteq A \implies v(R, A) \cap B \subseteq v(R, B) \tag{2}$$

- Ok, so if a country likes the centrist candidate the most from all the candidates, it also likes the centrist candidate most among them and, say, the far-right candidate.

- Yes, you are very quick - the Marquis took off a non-existing (more non-existing than him) hat from his head to show appreciation - For example, in la patrie-mère, we have two rounds of voting. In the first we choose the top candidate by plurality voting and make them go into a second round against the runner-up. What does this say?

- Hmmm... Well, either plurality does not satisfy the α property, or France is wasting money to organize the second round.

- Correct, no α. If α was satisfied, then the winner from the first round should automatically win in the second round as well, because people choose from a smaller set.[2]

- Veeery cool - one could see Laura started enjoying whatever this fantasy, dream, hallucination was.

- Yes, yes, we're almost there, two simple properties left. First, *Pareto-optimality*. This property simply says: If every voter prefers a Girondin over a Montagnard, the voting rule should NOT choose the Montagnard. I mean, why would you, if you have at least a Girondin, which everyone likes better, but maybe there is someone even worthier. Definitely not a Montagnard though.

As Laura nodded throughout this argument, Condorcet proceeded to the next one.

- And lastly - *majoritarian*. This *granularity* lever does not set any properties, rather it regulates what type of information the function "uses" when it selects the winners. In most countries, people only consider the smallest amount of information possible - what is the most preferred candidate of each voter. The voters don't even submit anything about other candidates. So you, for example, are now forced to vote for the moderate, because if you selected the progressive, as you really wanted to, there is no information on whether you prefer the moderate or the conservative. So if the allmighty polls are correct and the progressive has a low score and you voted for them, it is as if you didn't vote, because you have no influence on who is chosen amongst the other more popular candidates.

[2] Assuming their preferences stayed intact

- Exactly what I was thinking, why? Just why?

- Emmm, I don't know. So, if you collect more information during an election, as they do in Australia for ex., where they ask for the actual preference tuples I mentioned before, the function can use some or all of this information to generate an outcome. Actually, those tuples are more often mathematically expressed as relations - $\succ_i$ for voter i's preferences. For example, the preference relation of the first voter from our initial example would be expressed like this:

$$G \succ_i J, G \succ_i M, J \succ_i M \tag{3}$$

Although we prefer small letters for candidates.

- Now, back to majoritarian functions. These exclusively consider who the majority of voters preferred for all pairs of candidates and it is also usually expressed with a relation - $\succ_M$.

So it is important for a majoritarian function that more people prefer a than b among those two ($a \succ_M b$) and more people like c than a among these two ($c \succ_M a$). What's not important, is how many more voters like a more than b, or "by how much" does a voter like a more than b and so on.

Observe also that this relation is defined for all pairs of candidates, because every voters either prefers a to b or b to a, so we can just count the voters and see what the majority prefers.

- I see, and this combination of properties guarantee us the fairest election outcome?

- You're almost as far from the truth as the Montagnards, my friend. There are properties of all shapes and sizes, some are more strict or less strict - he points to the levers - some are just different, it depends what we want.

- But we don't know what we want?

- Yes, mostly people have no idea what I'm talking about.

- So we end up with a rule that barely satisfies anything?

- Yes! Yes! And there is some much to choose from, there is the Top Cycle, there is the Uncovered Set from Nikola, there is the Kemeny rule, everything you want. Well... - he scratches his head - almost.

- So each rule satisfies some sort of properties and not others? And this *Uncovered Set* satisfies the ones you explained previously?

- Yes!

- Why is that?

- Well, I am a bit short on time as my "magic" - he waved his hands in front of his face - only lasts around 10 minutes - but I'll prove a simple one for you and then you can think about the rest.

- Ok, but first - what is the Uncovered Set exactly?

- So, as it is majoritarian, it's easiest to explain via this majority relation. Nikola introduced a new term called *covering*. A candidate a *covers* another candidate b, if for all candidates c that prefer b over c - $b \succ_M c$, a also is majority preferred over c - $a \succ_M c$. And using this term he defined the rule - the winners are all candidates that are not covered by some other candidate.

- In other words, why choose a candidate b, if there is another candidate a which is also preferred against those who b is preferred against, and is possibly preferred against more candidates? - Exactly!

- Convoluted... - but I like it!

- And now: PROOF TIME! - the Marquis almost yelled. I'll prove to you that the Uncovered Set satisfies Pareto-optimality. Remember, Pareto-optimality was when if every voter preferred a over b, the voting rule would not choose b. Now, say we have 2 candidates, a and b, and every voter prefers a to b:

$$\forall i \in N, a \succ_i b \tag{4}$$

Note that because the voters' preferences are a tuple and thus ordered, the relation $\succ_i$ is a total order.
The fact that $\succ_i$ is a total order means that:

$$\forall c, b \succ_i c \implies a \succ_i c \tag{5}$$

In simple terms - if a voters ranks a higher than b and b higher than c, then a is obviously ranked higher than c.
Now, let c be some candidate, such that b is chosen against c by the majority of voters, i.e. $b \succ_M c$. Now, because of individual preferences being total orders, we can infer $\forall j \in N, b \succ_j c \implies a \succ_j c$. In other words, every voter who prefers b to c also prefers a to c and because the majority prefers b to c, the majority also prefers a to c, thus $a \succ_M c$.
So, for every candidate c which b is preferred against, a is also preferred against them.

- Oh, and this is exactly the definition of *covering*, right? - Exactly, a covers b in this scenario, thus b will not be chosen by the voting rule as it is not uncovered. And voilà, we proved that the Uncovered Set satisfies Pareto-optimality.

- Amazing!

- Ok, it's time for me to go and for you to drop that vote in the ballot box!

- Wait, 2 quick questions - wouldn't the rule that has all the levers all the way at the top be the fairest one? And why did you say "almost" when talking about shapes and sizes of voting rules?

- No time at all, but you still have 30 seconds in this room, while I deconstruct my "magic" - he waved his hands again - Try for yourself and you'll answer both questions.

Laura quickly pushed all levers to the top and pressed "Profit". The room got covered in red. The Statue right in front of them was glowing. The room started to quickly disappear, but Laura managed to read:

Kenneth Arrow, Arrow's Impossibility Principle
"There is no rule satisfying Pareto-optimality, α, β^{+3}, IIA^4, which is not a dictatorship."

[3] Strongest contraction consistency
[4] Not even the strongest monotonicity property

- So, as usual, there is no perfect solution - Laura said to herself as she looked around the voting station and dropped her ballot in the box.

At home, she read that all she learned was true and that there were much more questions which social choice tried to answer. For example, the outcome of the rules the Marquis mentioned were always sets, but what if we are choosing a president, and we cannot have two at the same time?

- Well, I'll either learn it in university or wait another four years, maybe this guy comes back and explains. By the way, the guy...

Wikipedia:

Marquis of Condorcet, known as Nicolas de Condorcet, was a French philosopher and mathematician. His ideas, including support for a liberal economy, free and equal public instruction, constitutional government, and equal rights for women and people of all races, have been said to embody the ideals of the Age of Enlightenment, of which he has been called the "last witness" and Enlightenment rationalism. Condorcet criticized the Constitution written by the Montagnards, after they took power and his version of the Constitution was not put to a vote. As a result, he was branded a traitor. He died in prison after a period of hiding from the French Revolutionary authorities.